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Constitutive equations for micropolar hyper-elastic materials

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ABSTRACT

In this paper, the concept of hyper-elasticity in the micropolar continuum theory is investigated. The restrictions on the fourth-order elasticity tensors are investigated. Using the representation theorems, a general form of constitutive equations for micropolar hyper-elastic isotropic materials is presented. As some special cases, generalizations of the neo-Hookean and Mooney-Rivlin type materials to the micropolar continuum theory are presented. The generalized constitutive equations reduce to those of the microplar linear elasticity theory when the deformations are infinitesimal. Also, Updated Lagrangian finite element formulations for the micropolar hyper-elastic materials are presented. Considering two planar examples, it is shown that an increase in the micropolar parameter results in the reduction of the deformation of the bodies. Also, it is shown that for a specimen with very small dimensions, e.g. in the micron level, the micropolar effects are more sensible. Furthermore, it is shown that the influence of the micropolar parameters is dependent not only on the size of the body, but also to its geometry and loading conditions. For the problems in which the deformation is very close to a homogeneous state, the micropolar effects are negligible.

1. Introduction

A micropolar medium is a classical continuum in which each particle is associated with a micro-structure. The micro-structure is in fact another continuum capable of undergoing only rigid rotations (Eringen and Kafadar, 1976). The classical and micropolar continuum theories have some differences from both kinematic and kinetic view points. From kinematic point of view, in the classical continuum mechanics the motion of material particles are described by the position vectors identifying the location of each particle as a function of time (Rubin, 2000). At each particle of a micropolar continuum, there is a micro-structure which can rotate independently from the surrounding medium. So, every particle contains six degrees of freedom, three translational which are assigned to the hyper-elasticity and three rotational ones which are referred to the micro-structure. In the classical continuum theory, from the kinetic point of view, the effect of a surface element on a neighboring one is expressible by only a traction vector. In the micropolar theory, the interaction between two adjacent surface elements is considered via a couple vector in addition to the traction vector (Eringen, 1968).

The beginning of the rational theories of the polar continua goes back to E. and F. Cosserat in 1909. In the 50s and 60s, extensive developments have been done by several authors. It is not our purpose to give a detailed historical exposition of the subject. An account of the historical development, as well as various contributions have been cited by Truesdell and Toupin (1960), Truesdell and Noll (1965) and Eringen and Kafadar (1976). Among them, the essential developments in the field of micropolar theory are due to Eringen and his co-workers. They completed the mathematical foundations of the micropolar continuum by 1971. Then, several researchers began to solve many problems based on the infinitesimal deformation of the micropolar media. Many of the solutions obtained in linear micropolar elasticity have been presented by Nowaki (1976) and Dyszlewicz (2004).

Recently, renewed interest in the micropolar continuum theory has been arised mainly for the context of localization computations. Today it is fairly well understood that numerical modeling of materials exhibiting strain softening, or non-symmetric material operators due to non-associated flow rules, for example, leads to a pathological mesh dependence of the post-peak response within the classical continuum description when the deformation pattern obeys a highly localized zone (Steinmann, 1994). Researchers looking for a remedy for this deficiency revived the micropolar approach. Here, we only refer to the works of Mühlhaus and Vardoulakis (1987), Steinmann and Willam (1991), de Borst (1993), Iordache and Willam (1998) and Sharbati and Naghdabadi (2006). In the context of micropolar finite elastic–plastic deformations, Steinmann (1994) developed a comprehensive theory based on micropolar hyper-elasticity and generalization of the so-called multiplicative plasticity in a modern geometry fasion. Later, Forest

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and Sievert (2003), Grammenoudis and Tsakmakis (2001, 2007) and Grammenoudis et al. (2007) investigated the micropolar multiplicative elastic–plastic theory in a thermodynamic framework with more detail. The theory of hypo-plasticity has been presented and numerically implemented on localization analysis of the granular materials by Tejchman (2004) and Tejchman and Wu (2007).

In the constitutive theory of continuous media, the polynomial forms of the tensor functions are often used. In order to present such polynomial expressions, we need to use representation theorems of tensor functions. The foundations of such theorems have been well stablished through 1955–1994 by several scientists. Among them, we refer to the works by Spencer (1971), Wang (1970) and Zheng (1994).

When the work done in elastic deformation is stored as internal energy, so that the stresses are derivable from a stored-energy function as a potential, the material is called hyper-elastic or Green elastic (Truesdell and Noll, 1965). The generalization of the concept of hyper-elasticity to the polar media has been performed by several authors. Here, we refer to the pioneer works of Eringen and Suhubi (1964), Toupin (1964) and Kafadar and Eringen (1971). In the recent literature, a fine treatment of the subject is given by Steinmann (1994), Forest and Sievert (2003) and Grammenoudis and Tsakmakis (2007). In this theory, stress and couple stress tensors, as well as entropy are derivable from a stored energy function.

In this paper, at first some basic relations of the micropolar continuum mechanics used in the next sections are presented. Based on the representation theorems proposed by Zheng (1994), the general form of the stress and couple stress tensors in the hyper-elastic type constitutive equations are derived. The rate form of the constitutive equations with material and spatial descriptions are derived. The integrability conditions for the given elasticities are investigated. As some special cases, the constitutive equations of neo-Hookean and Mooney-Rivlin materials are generalized to the micropolar continuum theory.

2. Basic relations of the micropolar continuum mechanics

In this section, we briefly present some basic relations of the micropolar continuum theory which are essential for the next sections. For more details and discussions, we refer to the works of Eringen (1968), Kafadar and Eringen (1971) and Eringen and Kafadar (1976). All Latin indices, e.g. i and K take only the values 1, 2 and 3. Latin indices with small and capital letters refer, respectively, to spatial and material coordinate systems. Greek indices, e.g. υ and ϑ do not obey a general rule and take the specified values defined in the corresponding equations. However, the summation convention holds for all repeated indices of Latin or Greek ones.

2.1. Kinematics

Let $\mathscr{B} \subset \mathbb{R}^3$ be the reference configuration of a continuum body at time $t = t_0$. We show the center of the macro-element at each point by $\mathbf{Z} \in \mathscr{B}$. A smooth macro-deformation is a one-to-one mapping $\chi : \mathscr{B} \to \mathscr{S} \subset \mathbb{R}^3$. We refer to $\mathbf{z} \in \mathscr{S}$ as a point in the current configuration $\mathscr{S} = \chi(\mathscr{B})$. In other words, at time t the material particle located at $\mathbf{Z}$ goes to the spatial position $\mathbf{z}$ in space. Similarly, $\mathbf{z}$ is the center of the macro-element at the same point in the current configuration. Now consider two systems of curvilinear coordinates X^I and x^i in the undeformed and deformed configurations, respectively. Let $\mathbf{G}_I$ and $\mathbf{g}_i$ be covariant base vectors tangent to the coordinate curves X^I and x^i, and $\mathbf{G}^I$ and $\mathbf{g}^i$ be their corresponding contravariant base vectors, respectively. Definition of the curvilinear coordinates $\mathbf{X}$ and $\boldsymbol{x}$ allows us to write $\mathbf{X} = \mathbf{X}(\mathbf{Z})$, $\mathbf{Z} = \mathbf{Z}(\mathbf{X})$, $\boldsymbol{x} = \boldsymbol{x}(\mathbf{z})$ and $\mathbf{z} = \mathbf{z}(\boldsymbol{x})$ which yield $\boldsymbol{x} = \boldsymbol{x}(\mathbf{z}(\mathbf{Z}(\mathbf{X})))$, or, for short, $\boldsymbol{x} = \boldsymbol{x}(\mathbf{X})$ and $\mathbf{X} = \mathbf{X}(\boldsymbol{x})$ (Truesdell and Toupin, 1960). The deformation gradient is the derivative of the deformation χ in the following form:

$$\mathbf{F}(\mathbf{X}, t) = D\chi = \frac{\partial \chi}{\partial \mathbf{X}} = \frac{\partial x}{\partial \mathbf{X}} = x^i_{,I}\mathbf{g}_i \otimes \mathbf{G}^I, \quad J = \det(\mathbf{F}) > 0, \tag{1}$$

where comma denotes differentiation with respect to the coordinates. Each point of $\mathscr{B}$ contains a micro-structure which has only rigid rotations. Angle of rotation of the micro-structure constitute a rotation (pseudo) vector $\boldsymbol{\varphi}$ with magnitude θ. The proper orthogonal micro-rotation tensor $\overline{\mathbf{R}}$ corresponding to $\boldsymbol{\varphi}$ is in the following form (Eringen and Kafadar, 1976; Ramezani and Naghdabadi, 2007):

$$\begin{aligned}\overline{\mathbf{R}} &= \overline{R}^i_{\ J}\mathbf{g}_i \otimes \mathbf{G}^J = \exp(\boldsymbol{\Phi})\bar{\mathbf{g}} \\ &= \left(\mathbf{I} + \frac{\sin\theta}{\theta}\boldsymbol{\Phi} + \frac{1-\cos\theta}{\theta^2}\boldsymbol{\Phi}\boldsymbol{\Phi}\right)\bar{\mathbf{g}}, \quad \det(\overline{\mathbf{R}}) = 1,\end{aligned} \tag{2}$$

where $\boldsymbol{\Phi}$ is the skew-symmetric tensor corresponding to $\boldsymbol{\varphi}$, $\mathbf{I}$ is the identity tensor and $\bar{\mathbf{g}} = \bar{g}^i_{\ J}\mathbf{g}_i \otimes \mathbf{G}^J$ with $\bar{g}^i_{\ J} = \mathbf{g}^i.\mathbf{G}_J$ is a shifter. Now consider an internal point in the micro-structure located at the position $\boldsymbol{\Xi} = \Xi^K\mathbf{G}_K$ with respect to the center $\mathbf{Z}$. In the micropolar theory, it is assumed that $\boldsymbol{\Xi}$ transforms to the vector $\xi = \xi^i\mathbf{g}_i$ under the rigid rotation $\overline{\mathbf{R}}$ and by the relation $\xi = \overline{\mathbf{R}}\boldsymbol{\Xi}$ (Eringen and Kafadar, 1976). The spin tensor of the micro-structure, $\overline{\boldsymbol{\Omega}}$, can then be obtained as follows:

$$\dot{\xi} = \dot{\overline{\mathbf{R}}}\boldsymbol{\Xi} = (\dot{\overline{\mathbf{R}}}\overline{\mathbf{R}}^T)\xi = \overline{\boldsymbol{\Omega}}\xi, \quad \overline{\boldsymbol{\Omega}} = \dot{\overline{\mathbf{R}}}\overline{\mathbf{R}}^T = -\overline{\boldsymbol{\Omega}}^T. \tag{3}$$

The spin tensor $\overline{\boldsymbol{\Omega}}$ is called gyration tensor. Now, the angular velocity vector $\bar{\omega}$ of the micro-structure can be constructed from Eqs. (2) and (3) as follows:

$$\begin{aligned}\bar{\omega} &= -\frac{1}{2}\boldsymbol{\varepsilon} : \overline{\boldsymbol{\Omega}} = \boldsymbol{\Lambda}\dot{\boldsymbol{\varphi}}, \\ \boldsymbol{\Lambda} &= \frac{\sin\theta}{\theta}\mathbf{I} + \frac{1}{\theta^2}\left(1 - \frac{\sin\theta}{\theta}\right)\boldsymbol{\varphi} \otimes \boldsymbol{\varphi} + \frac{\cos\theta - 1}{\theta^2}\boldsymbol{\varepsilon}\boldsymbol{\varphi},\end{aligned} \tag{4}$$

where (:) stands for the double contraction operation.

The motivation for the choice of the micropolar strain measures is the definition of the classical symmetric positive definite right and left stretch tensors $\mathbf{U} = \mathbf{R}^T\mathbf{F}$ and $\mathbf{V} = \mathbf{F}\mathbf{R}^T$, respectively, where $\mathbf{R}$ is a proper orthogonal tensor. A similar idea (with a slightly difference) is used for defining the material and spatial Cosserat deformation tensors, $\overline{\mathbf{U}}$ and $\overline{\mathbf{V}}$ as follows (Kafadar and Eringen, 1971):

$$\begin{aligned}&\mathbf{F} = \overline{\mathbf{R}}\overline{\mathbf{U}}^{\mathbf{T}} = \overline{\mathbf{V}}\overline{\mathbf{R}} \Rightarrow \overline{\mathbf{U}} = \mathbf{F}^{\mathbf{T}}\overline{\mathbf{R}}, \quad \overline{\mathbf{V}} = \overline{\mathbf{R}}\overline{\mathbf{U}}^{\mathbf{T}}\overline{\mathbf{R}}^{\mathbf{T}} = \mathbf{F}\overline{\mathbf{R}}^{\mathbf{T}}, \\ &\overline{\mathbf{U}}\overline{\mathbf{U}}^{\mathbf{T}} = \mathbf{U}^{\mathbf{2}} = \mathbf{C}, \quad \overline{\mathbf{V}}\overline{\mathbf{V}}^{\mathbf{T}} = \mathbf{V}^{\mathbf{2}} = \mathbf{B}, \quad \det\overline{\mathbf{U}} = \det\overline{\mathbf{V}} = \det\mathbf{F} = \mathbf{J} > \mathbf{0},\end{aligned} \tag{5}$$

where $\mathbf{C}$ and $\mathbf{B}$ are the classical symmetric positive definite right and left Cauchy–Green deformation tensors, respectively. Also, the material and spatial Cosserat wryness tensors, $\boldsymbol{\Gamma}$ and $\boldsymbol{\gamma}$ have the following forms (Eringen and Kafadar, 1976):

$$\boldsymbol{\Gamma} = \frac{1}{2}\boldsymbol{\varepsilon} : (\overline{\mathbf{R}}^T\nabla_{\mathbf{X}}\overline{\mathbf{R}}), \quad \boldsymbol{\gamma} = \overline{\mathbf{R}}\boldsymbol{\Gamma}\overline{\mathbf{R}}^{\mathbf{T}}, \quad \Gamma_{\mathbf{LK}} = \frac{\mathbf{1}}{\mathbf{2}}\varepsilon_{\mathbf{LMN}}\overline{\mathbf{R}}^{\mathbf{M}}_{\mathbf{p}}\overline{\mathbf{R}}^{\mathbf{pN}}_{|\mathbf{K}}, \tag{6}$$

where vertical bar (|) denotes covariant differentiation, and $\nabla_{\mathbf{X}}$ is the covariant differentiation operator defined in the coordinate system located at the reference configuration. The material and spatial micropolar strain tensors are defined as

$$\overline{\mathbf{H}} = \overline{\mathbf{U}} - \mathbf{I}, \quad \bar{\mathbf{h}} = \mathbf{I} - \overline{\mathbf{V}}^{-\mathbf{T}}. \tag{7}$$

For the case of infinitesimal deformations, the components of the infinitesimal strain and wryness tensors, $\bar{\mathbf{e}}$ and $\boldsymbol{\kappa}$, take the following forms (Eringen and Kafadar, 1976):

$$\overline{\mathbf{H}} \approx \bar{\mathbf{h}} \approx \bar{\mathbf{e}} = (\nabla\mathbf{u})^T + \boldsymbol{\Phi}, \quad \boldsymbol{\Gamma} \approx \boldsymbol{\kappa} = (\nabla\boldsymbol{\varphi})^T. \tag{8}$$

2.2. Micropolar stress tensor, couple stress tensor and balance laws

In the micropolar theory, the interaction between two adjacent surface elements is considered via a couple vector in addition to the traction vector. The following relations hold between the traction vector $\mathbf{t}^{(\mathbf{n})}$, the stress tensor $\bar{\boldsymbol{\sigma}}$, the couple vector $\mathbf{m}^{(\mathbf{n})}$, the couple stress tensor $\mathbf{m}$, and the unit normal vector to the surface $\mathbf{n}$ (Eringen, 1968):

$$\mathbf{t}^{(\mathbf{n})} = \mathbf{t}^i n_i = \mathbf{n}.\bar{\boldsymbol{\sigma}} = \bar{\sigma}^{ij} n_i \mathbf{g}_j, \quad \mathbf{m}^{(\mathbf{n})} = \mathbf{m}^i n_i = \mathbf{n}.\mathbf{m} = m^{ij} n_i \mathbf{g}_j. \tag{9}$$

Also, we define the micropolar first Piola–Kirchhoff stress and couple stress tensors $\overline{\mathbf{P}}$ and $\mathbf{M}$, and the micropolar material stress and couple stress tensors $\boldsymbol{\Sigma}_{(1)}$ and $\mathbf{M}_{(1)}$, respectively, as follows (Ramezani and Naghdabadi, 2007):

$$\{\overline{\mathbf{P}}, \mathbf{M}\} J \mathbf{F}^{-1}\{\bar{\boldsymbol{\sigma}}, \mathbf{m}\}, \quad \{\boldsymbol{\Sigma}_{(1)}, \mathbf{M}_{(1)}\} = \{\overline{\mathbf{P}}, \mathbf{M}\}\overline{\mathbf{R}}. \tag{10}$$

The material form of balance of linear momentum and moment of momentum in the micropolar media have the following local forms (Eringen and Kafadar, 1976):

$$\mathrm{Div}\overline{\mathbf{P}} + \rho_0 \mathbf{f} = \rho_0 \mathbf{a}, \quad \bar{P}^{Ji}_{|J} + \rho_0 f^i = \rho_0 a^i, \tag{11}$$

$$\mathrm{Div}\mathbf{M} + \boldsymbol{\varepsilon} : (\mathbf{F}\overline{\mathbf{P}}) + \rho_0 \mathbf{l} = \rho_0 \dot{\mathbf{h}}_{int}, \quad M^{Ki}_{|K} + \varepsilon^{i}_{lm.} x^l_{,K} \bar{P}^{Km} + \rho_0 l^i = \rho_0 \dot{h}^i_{int}, \tag{12}$$

where $\mathbf{f} = \mathbf{f}(\mathbf{X},t)$ is the body force density vector, $\mathbf{a}$ is the acceleration vector, $\mathbf{h}_{int}$ is density of the internal angular momentum vector, $\mathbf{l}$ is the body couple density vector and ρ_0 is the referential density. For a purely mechanical process, the power of deformation per unit reference volume is as follows (Ramezani and Naghdabadi, 2007):

$$P = \rho_0 \dot{\psi} = J[\bar{\boldsymbol{\sigma}} : \overline{\mathbf{D}} + \mathbf{m} : \boldsymbol{\Psi}] = \boldsymbol{\Sigma}_{(1)} : \dot{\overline{\mathbf{U}}} + \mathbf{M}_{(1)} : \dot{\boldsymbol{\Gamma}}^T \tag{13}$$

with ψ as the Helmholtz free energy function and

$$\overline{\mathbf{D}} = \mathbf{L}^T + \overline{\boldsymbol{\Omega}}, \quad \mathbf{L} = \nabla_x \mathbf{v}, \quad \boldsymbol{\Psi} = (\nabla_x \bar{\omega})^T, \tag{14}$$

where $\nabla_{\mathbf{x}}$ is the covariant differentiation operator defined in the coordinate system located at the current configuration. Material time derivatives of $\overline{\mathbf{U}}$ and $\boldsymbol{\Gamma}$ are in the following forms (Ramezani and Naghdabadi, 2007):

$$\dot{\overline{\mathbf{U}}} = \mathbf{F}^T \overline{\mathbf{D}}\overline{\mathbf{R}}, \quad \dot{\boldsymbol{\Gamma}} = \overline{\mathbf{R}}^T \boldsymbol{\Psi}^T \mathbf{F}. \tag{15}$$

Also, time differentiation of Eqs. $(5)_3$ and $(6)_2$ and using Eq. (15) results in

$$\dot{\overline{\mathbf{V}}} = \overline{\mathbf{D}}^T \overline{\mathbf{V}} + \overline{\boldsymbol{\Omega}}\overline{\mathbf{V}} - \overline{\mathbf{V}}\overline{\boldsymbol{\Omega}}, \quad \dot{\gamma} = \boldsymbol{\Psi}^{\mathbf{T}}\overline{\mathbf{V}} - \gamma\overline{\boldsymbol{\Omega}} + \overline{\boldsymbol{\Omega}}\gamma. \tag{16}$$

Based on Eq. (13), the pairs $(J\bar{\boldsymbol{\sigma}}, \overline{\mathbf{D}})$, $(\boldsymbol{\Sigma}_{(1)}, \dot{\overline{\mathbf{U}}})$, $(J\mathbf{m}, \boldsymbol{\Psi})$ and $(\mathbf{M}_{(1)}, \dot{\boldsymbol{\Gamma}}^T)$ are defined as the energy pairs in the micropolar continuum theory (Ramezani and Naghdabadi, 2007).

3. Micropolar hyper-elasticity

3.1. Rate-type micropolar hyper-elastic constitutive equations

In this section, we postulate that there is a stored energy function from which the stress and couple stress tensors can be derived by differentiating this function with respect to the deformation parameters. For a purely mechanical process, Eq. (13) results in

$$\boldsymbol{\Sigma}_{(1)} = \rho_0 \frac{\partial \hat{\psi}(\overline{\mathbf{U}}, \boldsymbol{\Gamma})}{\partial \overline{\mathbf{U}}}, \quad \mathbf{M}_{(1)} = \rho_0 \frac{\partial \hat{\psi}(\overline{\mathbf{U}}, \boldsymbol{\Gamma})}{\partial \boldsymbol{\Gamma}^T}. \tag{17}$$

Using chain rule of differentiation together with Eqs. (5), (6) and (10), one may obtain the following spatial form of the hyper-elasticitic constitutive equations for the isotropic micropolar materials (see also Kafadar and Eringen, 1971):

$$\bar{\boldsymbol{\tau}} = J\bar{\boldsymbol{\sigma}} = \rho_0 \overline{\mathbf{V}} \frac{\partial \tilde{\psi}(\overline{\mathbf{V}}, \gamma)}{\partial \overline{\mathbf{V}}^{\mathbf{T}}}, \quad \boldsymbol{\mu} = \mathbf{Jm} = \rho_{\mathbf{0}} \overline{\mathbf{V}} \frac{\partial \tilde{\psi}(\overline{\mathbf{V}}, \gamma)}{\partial \gamma^{\mathbf{T}}}. \tag{18}$$

It is noted that in Eqs. (17) and (18), the notations $\psi = \hat{\psi}(\overline{\mathbf{U}}, \boldsymbol{\Gamma}) = \tilde{\psi}(\overline{\mathbf{V}}, \gamma)$ have been used. Time differentiation of Eq. (17) with the aid of the chain rule of differentiation results in

$$\dot{\boldsymbol{\Sigma}}_{(1)} = \mathbf{A}_1 : \dot{\overline{\mathbf{U}}} + \mathbf{A}_2 : \dot{\boldsymbol{\Gamma}}^T, \quad \dot{\mathbf{M}}_{(1)} = \mathbf{A}_3 : \dot{\overline{\mathbf{U}}} + \mathbf{A}_4 : \dot{\boldsymbol{\Gamma}}^T, \tag{19}$$

where $\mathbf{A}_\upsilon$ $(\upsilon = 1,2,3,4)$ are the following fourth-order tensors:

$$\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4\} = \rho_0 \left\{ \frac{\partial^2 \hat{\psi}}{\partial \overline{\mathbf{U}} \partial \overline{\mathbf{U}}}, \frac{\partial^2 \hat{\psi}}{\partial \overline{\mathbf{U}} \partial \boldsymbol{\Gamma}^{\mathbf{T}}}, \frac{\partial^2 \hat{\psi}}{\partial \boldsymbol{\Gamma}^T \partial \overline{\mathbf{U}}}, \frac{\partial^2 \hat{\psi}}{\partial \boldsymbol{\Gamma}^T \partial \boldsymbol{\Gamma}^T} \right\} \tag{20}$$

and satisfy the following symmetry conditions:

$$\{A_1^{IJKL}, A_4^{IJKL}\} = \{A_1^{KLIJ}, A_4^{KLIJ}\}, A_2^{IJKL} = A_3^{KLIJ}. \tag{21}$$

Now, we define the objective second-order tensors $\bar{\boldsymbol{\tau}}^\nabla$ and $\boldsymbol{\mu}^\nabla$ in the following form:

$$\begin{Bmatrix} \bar{\boldsymbol{\tau}}^\nabla \\ \boldsymbol{\mu}^\nabla \end{Bmatrix} = \mathbf{F} \frac{d}{dt}\left(\mathbf{F}^{-1} \begin{Bmatrix} \bar{\boldsymbol{\tau}} \\ \boldsymbol{\mu} \end{Bmatrix} \overline{\mathbf{R}} \right) \overline{\mathbf{R}}^T = \mathbf{F} \begin{Bmatrix} \dot{\boldsymbol{\Sigma}}_{(1)} \\ \dot{\mathbf{M}}_{(1)} \end{Bmatrix} \overline{\mathbf{R}}^T = \begin{Bmatrix} \dot{\bar{\boldsymbol{\tau}}} \\ \dot{\boldsymbol{\mu}} \end{Bmatrix} - \mathbf{L} \begin{Bmatrix} \bar{\boldsymbol{\tau}} \\ \boldsymbol{\mu} \end{Bmatrix} + \begin{Bmatrix} \bar{\boldsymbol{\tau}} \\ \boldsymbol{\mu} \end{Bmatrix} \overline{\boldsymbol{\Omega}}. \tag{22}$$

Also, by defining $\{\bar{\boldsymbol{\sigma}}^\nabla, \mathbf{m}^\nabla\} = J^{-1}\{\bar{\boldsymbol{\tau}}^\nabla, \boldsymbol{\mu}^\nabla\}$ and recalling $\dot{J} = J\,tr(\mathbf{D})$, we may write

$$\begin{Bmatrix} \bar{\boldsymbol{\sigma}}^\nabla \\ \mathbf{m}^\nabla \end{Bmatrix} = \frac{1}{J} \begin{Bmatrix} \bar{\boldsymbol{\tau}}^\nabla \\ \boldsymbol{\mu}^\nabla \end{Bmatrix} = \begin{Bmatrix} \dot{\bar{\boldsymbol{\sigma}}} \\ \dot{\mathbf{m}} \end{Bmatrix} - \mathbf{L} \begin{Bmatrix} \bar{\boldsymbol{\sigma}} \\ \mathbf{m} \end{Bmatrix} + \begin{Bmatrix} \bar{\boldsymbol{\sigma}} \\ \mathbf{m} \end{Bmatrix} \overline{\boldsymbol{\Omega}} + tr(\mathbf{D}) \begin{Bmatrix} \bar{\boldsymbol{\sigma}} \\ \mathbf{m} \end{Bmatrix}. \tag{23}$$

The definitions of $\{\bar{\boldsymbol{\tau}}^\nabla, \boldsymbol{\mu}^\nabla\}$ and $\{\bar{\boldsymbol{\sigma}}^\nabla, \mathbf{m}^\nabla\}$ are very similar to the so-called Truesdell rate of the Kirchhoff stress $\boldsymbol{\tau} = J\boldsymbol{\sigma}$, and the Cauchy stress $\boldsymbol{\sigma}$ in the classical continuum theory, except that $\overline{\boldsymbol{\Omega}}$ is replaced for $-\mathbf{L}^T$.

For the case of isotropic micropolar materials, differentiating Eqs. (18) with respect to time and using Eqs. (16) and (22), we obtain

$$\bar{\boldsymbol{\tau}}^\nabla = \mathbf{b_1} : \overline{\mathbf{D}} + \mathbf{b_2} : \boldsymbol{\Psi}, \quad \boldsymbol{\mu}^\nabla = \mathbf{b_3} : \overline{\mathbf{D}} + \mathbf{b_4} : \boldsymbol{\Psi}, \tag{24}$$

where the fourth-order tensors $\mathbf{b}_\upsilon$ $(\upsilon = 1,2,3,4)$ have the following components:

$$\{b_1, b_2, b_3, b_4\}^{ijkl} = \bar{V}^i_{.m} \bar{V}^k_{.p} \left\{ \frac{\partial^2 \psi}{\partial \bar{V}_{lp} \partial \bar{V}_{jm}}, \frac{\partial^2 \psi}{\partial \gamma_{lp} \partial \bar{V}_{jm}}, \frac{\partial^2 \psi}{\partial \bar{V}_{lp} \partial \gamma_{jm}}, \frac{\partial^2 \psi}{\partial \gamma_{lp} \partial \gamma_{jm}} \right\}. \tag{25}$$

Similarly, combining Eqs. (23) and (24), we obtain

$$\begin{Bmatrix} \bar{\boldsymbol{\sigma}}^\nabla \\ \mathbf{m}^\nabla \end{Bmatrix} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \\ \mathbf{a}_3 & \mathbf{a}_4 \end{bmatrix} : \begin{Bmatrix} \overline{\mathbf{D}} \\ \boldsymbol{\Psi} \end{Bmatrix}, \qquad \mathbf{a}_\upsilon = J^{-1}\mathbf{b}_\upsilon, \quad \upsilon = 1,2,3,4. \tag{26}$$

The fourth-order tensors $\mathbf{a}_\upsilon$ and $\mathbf{b}_\upsilon$ satisfy the following symmetry conditions:

$$\{a,b\}_1^{ijkl} = \{a,b\}_1^{klij}, \quad \{a,b\}_4^{ijkl} = \{a,b\}_4^{klij}, \quad \{a,b\}_2^{ijkl} = \{a,b\}_3^{klij}. \tag{27}$$

In order to obtain the relation between $\mathbf{A}_\upsilon$ and $\mathbf{b}_\upsilon (\upsilon = 1,2,3,4)$, we combine Eqs. (15), (19), (22) and (24) to obtain

$$\mathbf{F}\left[\begin{Bmatrix} \mathbf{A}_1 \\ \mathbf{A}_3 \end{Bmatrix} : (\mathbf{F}^T \overline{\mathbf{D}}\overline{\mathbf{R}}) + \begin{Bmatrix} \mathbf{A}_2 \\ \mathbf{A}_4 \end{Bmatrix} : (\mathbf{F}^T \boldsymbol{\Psi} \overline{\mathbf{R}}) \right] \overline{\mathbf{R}}^{\mathbf{T}} = \begin{Bmatrix} \mathbf{b}_1 \\ \mathbf{b}_3 \end{Bmatrix} : \overline{\mathbf{D}} + \begin{Bmatrix} \mathbf{b}_2 \\ \mathbf{b}_4 \end{Bmatrix} : \boldsymbol{\Psi}, \tag{28}$$

which yields the following relations between the components of $\mathbf{b}_\upsilon$ and $\mathbf{A}_\upsilon$ for the case of isotropic hyper-elastic materials:

$$b_\upsilon^{ijkl} = x^i_{,I} \bar{R}^j_{.J} x^k_{,K} \bar{R}^l_{.L} A_\upsilon^{IJKL} \quad (\upsilon = 1,2,3,4) \tag{29}$$

or equivalently

$$A_{\upsilon}^{IJKL} = X_{,i}^{I}\bar{R}_{j}^{J}X_{,k}^{K}\bar{R}_{l}^{L}b_{\upsilon}^{ijkl} \quad (\upsilon = 1,2,3,4). \tag{30}$$

3.2. Restrictions on the elasticities

Following Simo and Pister (1984) in the context of the classical continuum, we want to obtain the conditions on $\mathbf{A}_{\upsilon}$ $(\upsilon = 1,2,3,4)$ under which solutions exist for ψ or $\mathbf{\Sigma}_{(1)}$ and $\mathbf{M}_{(1)}$. First, we consider the differential equation (17) and assume $\mathbf{\Sigma}_{(1)}$ and $\mathbf{M}_{(1)}$ are pre-determined quantities. Using Vainberg's theorem (Marsden and Hughes, 1983), the necessary and sufficient conditions for existing the energy function $\hat{\psi}(\overline{\mathbf{U}},\mathbf{\Gamma})$ are as follows:

$$\frac{\partial \Sigma_{(1)IJ}}{\partial \bar{U}_{KL}} = \frac{\partial \Sigma_{(1)KL}}{\partial \bar{U}_{IJ}}, \quad \frac{\partial \mathrm{M}_{(1)IJ}}{\partial \Gamma_{KL}} = \frac{\partial \mathrm{M}_{(1)KL}}{\partial \Gamma_{IJ}}, \tag{31}$$

which are the same as those presented in Eq. (21). In other words, if $\mathbf{\Sigma}_{(1)}$ and $\mathbf{M}_{(1)}$ have been specified, the necessary and sufficient for existing the energy function $\hat{\psi}$ are the symmetry conditions on $\mathbf{A}_{\upsilon}$ $(\upsilon = 1,4)$ in Eq. (21). Similar argument is valid for the existing the energy function $\tilde{\psi}$ when Eq. (27) hold. In the next step, consider the following material form differential equations:

$$\begin{aligned}
&\mathrm{d}\mathbf{\Sigma}_{(1)} = \mathbf{A}_1 : \mathrm{d}\overline{\mathbf{U}} + \mathbf{A}_2 : \mathrm{d}\mathbf{\Gamma}^{\mathbf{T}}, \quad \mathrm{d}\dot{\mathbf{M}}_{(1)} = \mathbf{A}_3 : \mathrm{d}\overline{\mathbf{U}} + \mathbf{A}_4 : \mathrm{d}\mathbf{\Gamma}^{\mathbf{T}},\\
&\frac{\partial \mathbf{\Sigma}_{(1)}(\mathbf{X},\overline{\mathbf{U}},\mathbf{\Gamma})}{\partial \overline{\mathbf{U}}} = \mathbf{A}_1(\mathbf{X},\overline{\mathbf{U}},\mathbf{\Gamma}), \quad \frac{\partial \mathbf{\Sigma}_{(1)}(\mathbf{X},\overline{\mathbf{U}},\mathbf{\Gamma})}{\partial \mathbf{\Gamma}^{\mathbf{T}}} = \mathbf{A}_2(\mathbf{X},\overline{\mathbf{U}},\mathbf{\Gamma}),\\
&\frac{\partial \mathbf{M}_{(1)}(\mathbf{X},\overline{\mathbf{U}},\mathbf{\Gamma})}{\partial \overline{\mathbf{U}}} = \mathbf{A}_3(\mathbf{X},\overline{\mathbf{U}},\mathbf{\Gamma}), \quad \frac{\partial \mathbf{M}_{(1)}(\mathbf{X},\overline{\mathbf{U}},\mathbf{\Gamma})}{\partial \mathbf{\Gamma}^{\mathbf{T}}} = \mathbf{A}_4(\mathbf{X},\overline{\mathbf{U}},\mathbf{\Gamma}).
\end{aligned} \tag{32}$$

In other words, assuming that $\mathbf{A}_{\upsilon}$ $(\upsilon = 1,2,3,4)$ as some given fourth-order tensors, we want to find the conditions under which the solutions for $\mathbf{\Sigma}_{(1)}$ and $\mathbf{M}_{(1)}$ can be found by solving differential equation (32). Again, Vainberg's theorem implies the necessary and sufficient conditions for such integrability as follows:

$$\begin{aligned}
&\frac{\partial A_1^{IJKL}}{\partial \Gamma_{NM}} = \frac{\partial A_2^{IJMN}}{\partial \bar{U}_{KL}}, \quad \frac{\partial A_3^{IJKL}}{\partial \Gamma_{NM}} = \frac{\partial A_4^{IJMN}}{\partial \bar{U}_{KL}}, \quad \frac{\partial A_1^{IJKL}}{\partial \bar{U}_{MN}} = \frac{\partial A_1^{IJMN}}{\partial \bar{U}_{KL}},\\
&\frac{\partial A_2^{IJKL}}{\partial \Gamma_{MN}} = \frac{\partial A_2^{IJMN}}{\partial \Gamma_{KL}}, \quad \frac{\partial A_3^{IJKL}}{\partial \bar{U}_{MN}} = \frac{\partial A_3^{IJMN}}{\partial \bar{U}_{KL}}, \quad \frac{\partial A_4^{IJKL}}{\partial \Gamma_{MN}} = \frac{\partial A_4^{IJMN}}{\partial \Gamma_{KL}}.
\end{aligned} \tag{33}$$

Such conditions may be applied on the spatial elasticities b_{υ}^{ijkl} (or a_{υ}^{ijkl}) by combining Eq. $(5)_3$, $(6)_3$, (30) and (33) in the following form:

$$\begin{aligned}
&\frac{\partial b_1^{ijkl}}{\partial \gamma_{nm}} = \frac{\partial b_2^{ijmn}}{\partial \bar{V}_{kl}}, \quad \frac{\partial b_3^{ijkl}}{\partial \gamma_{mn}} = \frac{\partial b_4^{ijmn}}{\partial \bar{V}_{kl}}, \quad \frac{\partial b_1^{ijkl}}{\partial \bar{V}_{mn}} = \frac{\partial b_1^{ijmn}}{\partial \bar{V}_{kl}},\\
&\frac{\partial b_2^{ijkl}}{\partial \gamma_{mn}} = \frac{\partial b_2^{ijmn}}{\partial \gamma_{kl}}, \quad \frac{\partial b_3^{ijkl}}{\partial \bar{V}_{mn}} = \frac{\partial b_3^{ijmn}}{\partial \bar{V}_{kl}}, \quad \frac{\partial b_4^{ijkl}}{\partial \gamma_{mn}} = \frac{\partial b_4^{ijmn}}{\partial \gamma_{kl}}.
\end{aligned} \tag{34}$$

It is noted that when the energy function ψ is specified, the conditions presented in Eqs. (31), (33) and (34) are identically satisfied. Such conditions have to be checked when the elasticities A_{υ}^{IJKL} or b_{υ}^{ijkl} are pre-determined quantities without any reference to a specified energy function.

3.3. General form of ψ for the isotropic micropolar materials

For the case of isotropic micropolar materials, ψ is a scalar-valued tensor function of the scalar invariants of $(\overline{\mathbf{V}},\gamma)$ (Kafadar and Eringen, 1971). Following the method proposed by Kafadar and Eringen (1971) and Zheng (1994), we decompose the non-symmetric tensors $\overline{\mathbf{V}}$ and γ to their symmetric and skew-symmetric parts as $\{\overline{\mathbf{V}}_{\mathbf{s}},\gamma_{\mathbf{s}}\} = \mathbf{sym}\{\overline{\mathbf{V}},\gamma\}$ **and** $\{\overline{\mathbf{V}}_{\mathbf{a}},\gamma_{\mathbf{a}}\} = \mathbf{skew}\{\overline{\mathbf{V}},\gamma\}$. From Zheng (1994), the resulting tensors $(\overline{\mathbf{V}}_{\mathbf{s}},\overline{\mathbf{V}}_{\mathbf{a}},\gamma_{\mathbf{s}},\gamma_{\mathbf{a}})$ have 39 scalar joint invariants which are trace of the following tensors:

$$\begin{aligned}
&\overline{\mathbf{V}}_{\mathbf{s}}, \overline{\mathbf{V}}_{\mathbf{s}}^2, \overline{\mathbf{V}}_{\mathbf{s}}^3, \overline{\mathbf{V}}_{\mathbf{a}}^2, \gamma_{\mathbf{s}}, \gamma_{\mathbf{s}}^2, \gamma_{\mathbf{s}}^3, \gamma_{\mathbf{a}}^2, \overline{\mathbf{V}}_{\mathbf{s}}\gamma_{\mathbf{s}}, \overline{\mathbf{V}}_{\mathbf{s}}^2\gamma_{\mathbf{s}}, \overline{\mathbf{V}}_{\mathbf{s}}\gamma_{\mathbf{s}}^2, \overline{\mathbf{V}}_{\mathbf{s}}^2\gamma_{\mathbf{s}}^2, \overline{\mathbf{V}}_{\mathbf{s}}\overline{\mathbf{V}}_{\mathbf{a}}^2, \overline{\mathbf{V}}_{\mathbf{s}}^2\overline{\mathbf{V}}_{\mathbf{a}}^2,\\
&\overline{\mathbf{V}}_{\mathbf{s}}^2\overline{\mathbf{V}}_{\mathbf{a}}^2\overline{\mathbf{V}}_{\mathbf{s}}\overline{\mathbf{V}}_{\mathbf{a}}, \overline{\mathbf{V}}_{\mathbf{s}}\gamma_{\mathbf{a}}^2, \overline{\mathbf{V}}_{\mathbf{s}}^2\gamma_{\mathbf{a}}^2, \overline{\mathbf{V}}_{\mathbf{s}}^2\gamma_{\mathbf{a}}^2\overline{\mathbf{V}}_{\mathbf{s}}\gamma_{\mathbf{a}}, \gamma_{\mathbf{s}}\overline{\mathbf{V}}_{\mathbf{a}}^2, \gamma_{\mathbf{s}}^2\overline{\mathbf{V}}_{\mathbf{a}}^2, \gamma_{\mathbf{s}}^2\overline{\mathbf{V}}_{\mathbf{a}}^2\gamma_{\mathbf{s}}\overline{\mathbf{V}}_{\mathbf{a}}, \gamma_{\mathbf{s}}\gamma_{\mathbf{a}}^2, \gamma_{\mathbf{s}}^2\gamma_{\mathbf{a}}^2,\\
&\gamma_{\mathbf{s}}^2\gamma_{\mathbf{a}}^2\gamma_{\mathbf{s}}\gamma_{\mathbf{a}}, \overline{\mathbf{V}}_{\mathbf{a}}\gamma_{\mathbf{a}}, \overline{\mathbf{V}}_{\mathbf{s}}\gamma_{\mathbf{s}}\overline{\mathbf{V}}_{\mathbf{a}}, \overline{\mathbf{V}}_{\mathbf{s}}^2\gamma_{\mathbf{s}}\overline{\mathbf{V}}_{\mathbf{a}}, \overline{\mathbf{V}}_{\mathbf{s}}\gamma_{\mathbf{s}}^2\overline{\mathbf{V}}_{\mathbf{a}}, \overline{\mathbf{V}}_{\mathbf{s}}\overline{\mathbf{V}}_{\mathbf{a}}^2\gamma_{\mathbf{s}}\overline{\mathbf{V}}_{\mathbf{a}}, \overline{\mathbf{V}}_{\mathbf{s}}\gamma_{\mathbf{s}}\gamma_{\mathbf{a}}, \overline{\mathbf{V}}_{\mathbf{s}}^2\gamma_{\mathbf{s}}\gamma_{\mathbf{a}},\\
&\overline{\mathbf{V}}_{\mathbf{s}}\gamma_{\mathbf{s}}^2\gamma_{\mathbf{a}}, \overline{\mathbf{V}}_{\mathbf{s}}\gamma_{\mathbf{a}}^2\gamma_{\mathbf{s}}\gamma_{\mathbf{a}}, \overline{\mathbf{V}}_{\mathbf{s}}\overline{\mathbf{V}}_{\mathbf{a}}\gamma_{\mathbf{a}}, \overline{\mathbf{V}}_{\mathbf{s}}\overline{\mathbf{V}}_{\mathbf{a}}^2\gamma_{\mathbf{a}}, \overline{\mathbf{V}}_{\mathbf{s}}\overline{\mathbf{V}}_{\mathbf{a}}\gamma_{\mathbf{a}}^2, \gamma_{\mathbf{s}}\overline{\mathbf{V}}_{\mathbf{a}}\gamma_{\mathbf{a}}, \gamma_{\mathbf{s}}\overline{\mathbf{V}}_{\mathbf{a}}^2\gamma_{\mathbf{a}}, \gamma_{\mathbf{s}}\overline{\mathbf{V}}_{\mathbf{a}}\gamma_{\mathbf{a}}^2.
\end{aligned} \tag{35}$$

Alternatively, one may construct scalar invariants J_{υ} $(\upsilon = 1,2,\ldots,39)$ in terms of $(\overline{\mathbf{V}},\overline{\mathbf{V}}^{\mathbf{T}},\gamma,\gamma^{\mathbf{T}})$ as follows:

$$\begin{aligned}
&J_1 = tr(\overline{\mathbf{V}}) \quad J_2 = tr(\overline{\mathbf{V}}^2) \quad J_3 = tr(\overline{\mathbf{V}}^3)\\
&J_4 = tr(\overline{\mathbf{V}}\overline{\mathbf{V}}^T) \quad J_5 = tr(\overline{\mathbf{V}}^2\overline{\mathbf{V}}^T) \quad J_6 = tr(\overline{\mathbf{V}}^2\overline{\mathbf{V}}^{2T})\\
&J_7 = tr(\overline{\mathbf{V}}^2\overline{\mathbf{V}}^{2T}\overline{\mathbf{V}}\overline{\mathbf{V}}^T) \quad J_8 = tr(\overline{\mathbf{V}}\gamma) \quad J_9 = tr(\overline{\mathbf{V}}\gamma^T)\\
&J_{10} = tr(\overline{\mathbf{V}}\gamma^2) \quad J_{11} = tr(\overline{\mathbf{V}}\gamma^{2T}) \quad J_{12} = tr(\overline{\mathbf{V}}\gamma\gamma^T)\\
&J_{13} = tr(\overline{\mathbf{V}}\gamma^T\gamma) \quad J_{14} = tr(\overline{\mathbf{V}}\gamma^2\gamma^T) \quad J_{15} = tr(\overline{\mathbf{V}}\gamma^T\gamma^2)\\
&J_{16} = tr(\overline{\mathbf{V}}^2\gamma) \quad J_{17} = tr(\overline{\mathbf{V}}^2\gamma^T) \quad J_{18} = tr(\overline{\mathbf{V}}^2\gamma^2)\\
&J_{19} = tr(\overline{\mathbf{V}}^2\gamma^{2T}) \quad J_{20} = tr(\overline{\mathbf{V}}^2\gamma\gamma^T) \quad J_{21} = tr(\overline{\mathbf{V}}^2\gamma^T\gamma)\\
&J_{22} = tr(\overline{\mathbf{V}}^T\overline{\mathbf{V}}\gamma) \quad J_{23} = tr(\overline{\mathbf{V}}\overline{\mathbf{V}}^T\gamma) \quad J_{24} = tr(\overline{\mathbf{V}}^T\overline{\mathbf{V}}\gamma^2)\\
&J_{25} = tr(\overline{\mathbf{V}}\overline{\mathbf{V}}^T\gamma^2) \quad J_{26} = tr(\overline{\mathbf{V}}^T\overline{\mathbf{V}}\gamma^T\gamma) \quad J_{27} = tr(\overline{\mathbf{V}}^2\gamma^{2T}\overline{\mathbf{V}}\gamma^T)\\
&J_{28} = tr(\overline{\mathbf{V}}^2\gamma^2\overline{\mathbf{V}}^T\gamma^T) \quad J_{29} = tr(\overline{\mathbf{V}}^2\overline{\mathbf{V}}^T\gamma) \quad J_{30} = tr(\overline{\mathbf{V}}^2\overline{\mathbf{V}}^T\gamma^T)\\
&J_{31} = tr(\overline{\mathbf{V}}^2\overline{\mathbf{V}}^{2T}\gamma) \quad J_{32} = tr(\overline{\mathbf{V}}\gamma^2\gamma^{2T}) \quad J_{33} = tr(\gamma)\\
&J_{34} = tr(\gamma^2) \quad J_{35} = tr(\gamma^3) \quad J_{36} = tr(\gamma\gamma^T)\\
&J_{37} = tr(\gamma^2\gamma^T) \quad J_{38} = tr(\gamma^2\gamma^{2T}) \quad J_{39} = tr(\gamma^2\gamma^{2T}\gamma\gamma^T)
\end{aligned} \tag{36}$$

In other words we can write the energy function ψ as $\psi = \bar{\bar{\psi}}(J_1,J_2,\ldots,J_{39})$. Using Eqs. (18) and (36) we obtain

$$\bar{\boldsymbol{\tau}} = \alpha_{\upsilon}\overline{\mathbf{V}}\frac{\partial J_{\upsilon}}{\partial \overline{\mathbf{V}}^T}, \quad \boldsymbol{\mu} = \alpha_{\upsilon}\overline{\mathbf{V}}\frac{\partial J_{\upsilon}}{\partial \gamma^T}, \tag{37}$$

where $\alpha_{\upsilon} = \rho_0\partial\bar{\bar{\psi}}/\partial J_{\upsilon}$ $(\upsilon = 1,2,\ldots,39)$ are scalar-valued tensor functions of the 39 invariants listed in Eq. (36). Finally, Eq. (37) yield the following non-linear hyper-elastic constitutive equations for an isotropic micopolar medium:

$$\begin{aligned}
\bar{\tau} = \overline{\mathbf{V}}\{&\alpha_1\mathbf{I} + \mathbf{2}\alpha_2\overline{\mathbf{V}} + \mathbf{3}\alpha_3\overline{\mathbf{V}}^2 + \mathbf{2}\alpha_4\overline{\mathbf{V}}^{\mathbf{T}} + \alpha_5(\overline{\mathbf{V}}\overline{\mathbf{V}}^{\mathbf{T}} + \overline{\mathbf{V}}^{\mathbf{T}}\overline{\mathbf{V}} + \overline{\mathbf{V}}^{\mathbf{2T}})\\
&+ \mathbf{2}\alpha_6(\overline{\mathbf{V}}\overline{\mathbf{V}}^{\mathbf{2T}} + \overline{\mathbf{V}}^{\mathbf{2T}}\overline{\mathbf{V}}) + \mathbf{2}\alpha_7(\overline{\mathbf{V}}\overline{\mathbf{V}}^{\mathbf{2T}}\overline{\mathbf{V}}\overline{\mathbf{V}}^{\mathbf{T}} + \overline{\mathbf{V}}^{\mathbf{T}}\overline{\mathbf{V}}^2\overline{\mathbf{V}}^{\mathbf{2T}} + \overline{\mathbf{V}}^{\mathbf{2T}}\overline{\mathbf{V}}\overline{\mathbf{V}}^{\mathbf{T}}\overline{\mathbf{V}})\\
&+ \alpha_8\gamma + \alpha_9\gamma^{\mathbf{T}} + \alpha_{10}\gamma^2 + \alpha_{11}\gamma^{\mathbf{2T}} + \alpha_{12}\gamma\gamma^{\mathbf{T}} + \alpha_{13}\gamma^{\mathbf{T}}\gamma + \alpha_{14}\gamma^2\gamma^{\mathbf{T}} + \alpha_{15}\gamma^{\mathbf{T}}\gamma^2\\
&+ \alpha_{16}(\overline{\mathbf{V}}\gamma + \gamma\overline{\mathbf{V}}) + \alpha_{17}(\overline{\mathbf{V}}\gamma^{\mathbf{T}} + \gamma^{\mathbf{T}}\overline{\mathbf{V}}) + \alpha_{18}(\overline{\mathbf{V}}\gamma^2 + \gamma^2\overline{\mathbf{V}}) + \alpha_{19}(\overline{\mathbf{V}}\gamma^{\mathbf{2T}} + \gamma^{\mathbf{2T}}\overline{\mathbf{V}})\\
&+ \alpha_{20}(\overline{\mathbf{V}}\gamma\gamma^{\mathbf{T}} + \gamma\gamma^{\mathbf{T}}\overline{\mathbf{V}}) + \alpha_{21}(\overline{\mathbf{V}}\gamma^{\mathbf{T}}\gamma + \gamma^{\mathbf{T}}\gamma\overline{\mathbf{V}}) + \alpha_{22}(\gamma + \gamma^{\mathbf{T}})\overline{\mathbf{V}}^{\mathbf{T}}\\
&+ \alpha_{23}\overline{\mathbf{V}}^{\mathbf{T}}(\gamma + \gamma^{\mathbf{T}}) + \alpha_{24}(\gamma^2 + \gamma^{\mathbf{2T}})\overline{\mathbf{V}}^{\mathbf{T}} + \alpha_{25}\overline{\mathbf{V}}^{\mathbf{T}}(\gamma^2 + \gamma^{\mathbf{2T}}) + \mathbf{2}\alpha_{26}\gamma^{\mathbf{T}}\gamma\overline{\mathbf{V}}^{\mathbf{T}}\\
&+ \alpha_{27}(\overline{\mathbf{V}}\gamma^{\mathbf{2T}}\overline{\mathbf{V}}\gamma^{\mathbf{T}} + \gamma^{\mathbf{2T}}\overline{\mathbf{V}}\gamma^{\mathbf{T}}\overline{\mathbf{V}} + \gamma^{\mathbf{T}}\overline{\mathbf{V}}^2\gamma^{\mathbf{2T}}) + \alpha_{28}(\overline{\mathbf{V}}\gamma^2\overline{\mathbf{V}}^{\mathbf{T}}\gamma^{\mathbf{T}} + \gamma^2\overline{\mathbf{V}}^{\mathbf{T}}\gamma^{\mathbf{T}}\overline{\mathbf{V}} + \gamma^{\mathbf{2T}}\overline{\mathbf{V}}^{\mathbf{2T}}\gamma)\\
&+ \alpha_{29}(\overline{\mathbf{V}}\overline{\mathbf{V}}^{\mathbf{T}}\gamma + \overline{\mathbf{V}}^{\mathbf{T}}\gamma\overline{\mathbf{V}} + \overline{\mathbf{V}}^{\mathbf{2T}}\gamma^{\mathbf{T}}) + \alpha_{30}(\overline{\mathbf{V}}\overline{\mathbf{V}}^{\mathbf{T}}\gamma^{\mathbf{T}} + \overline{\mathbf{V}}^{\mathbf{T}}\gamma^{\mathbf{T}}\overline{\mathbf{V}} + \overline{\mathbf{V}}^{\mathbf{2T}}\gamma)\\
&+ \alpha_{31}[\overline{\mathbf{V}}\overline{\mathbf{V}}^{\mathbf{2T}}(\gamma + \gamma^{\mathbf{T}}) + \overline{\mathbf{V}}^{\mathbf{2T}}(\gamma + \gamma^{\mathbf{T}})\overline{\mathbf{V}}] + \alpha_{32}\gamma^2\gamma^{\mathbf{2T}}\},
\end{aligned} \tag{38}$$

$$\begin{aligned}
\boldsymbol{\mu} = \overline{\mathbf{V}}\{&\alpha_8\overline{\mathbf{V}} + \alpha_9\overline{\mathbf{V}}^{\mathbf{T}} + \alpha_{10}(\overline{\mathbf{V}}\gamma + \gamma\overline{\mathbf{V}}) + \alpha_{11}(\overline{\mathbf{V}}^{\mathbf{T}}\gamma + \gamma\overline{\mathbf{V}}^{\mathbf{T}}) + \alpha_{12}\gamma^{\mathbf{T}}(\overline{\mathbf{V}} + \overline{\mathbf{V}}^{\mathbf{T}})\\
&+ \alpha_{13}(\overline{\mathbf{V}} + \overline{\mathbf{V}}^{\mathbf{T}})\gamma^{\mathbf{T}} + \alpha_{14}(\gamma\gamma^{\mathbf{T}}\overline{\mathbf{V}} + \gamma^{\mathbf{T}}\overline{\mathbf{V}}\gamma + \gamma^{\mathbf{2T}}\overline{\mathbf{V}}^{\mathbf{T}}) + \alpha_{15}(\overline{\mathbf{V}}^{\mathbf{T}}\gamma^{\mathbf{2T}} + \gamma\overline{\mathbf{V}}\gamma^{\mathbf{T}}\\
&+ \overline{\mathbf{V}}\gamma^{\mathbf{T}}\gamma) + \alpha_{16}\overline{\mathbf{V}}^2 + \alpha_{17}\overline{\mathbf{V}}^{\mathbf{2T}} + \alpha_{18}(\gamma\overline{\mathbf{V}}^{\mathbf{2T}} + \overline{\mathbf{V}}^{\mathbf{2T}}\gamma) + \alpha_{19}(\overline{\mathbf{V}}^{\mathbf{2T}}\gamma + \gamma\overline{\mathbf{V}}^{\mathbf{2T}})\\
&+ \alpha_{20}\gamma^{\mathbf{T}}(\overline{\mathbf{V}}^2 + \overline{\mathbf{V}}^{\mathbf{2T}}) + \alpha_{21}(\overline{\mathbf{V}}^2 + \overline{\mathbf{V}}^{\mathbf{2T}})\gamma^{\mathbf{T}} + \alpha_{22}\overline{\mathbf{V}}^{\mathbf{T}}\overline{\mathbf{V}} + \alpha_{23}\overline{\mathbf{V}}\overline{\mathbf{V}}^{\mathbf{T}}\\
&+ \alpha_{24}(\gamma\overline{\mathbf{V}}^{\mathbf{T}}\overline{\mathbf{V}} + \overline{\mathbf{V}}^{\mathbf{T}}\overline{\mathbf{V}}\gamma) + \alpha_{25}(\gamma\overline{\mathbf{V}}\overline{\mathbf{V}}^{\mathbf{T}} + \overline{\mathbf{V}}\overline{\mathbf{V}}^{\mathbf{T}}\gamma) + \mathbf{2}\alpha_{26}\overline{\mathbf{V}}^{\mathbf{T}}\overline{\mathbf{V}}\gamma^{\mathbf{T}}\\
&+ \alpha_{27}(\overline{\mathbf{V}}^{\mathbf{2T}}\gamma\overline{\mathbf{V}}\gamma + \gamma\overline{\mathbf{V}}^{\mathbf{2T}}\gamma\overline{\mathbf{V}} + \overline{\mathbf{V}}\gamma^2\overline{\mathbf{V}}^{\mathbf{2T}}) + \alpha_{28}(\gamma\overline{\mathbf{V}}^{\mathbf{T}}\gamma^{\mathbf{T}}\overline{\mathbf{V}}^2 + \overline{\mathbf{V}}^{\mathbf{T}}\gamma^{\mathbf{T}}\overline{\mathbf{V}}^2\gamma\\
&+ \overline{\mathbf{V}}\gamma^{\mathbf{2T}}\overline{\mathbf{V}}^{\mathbf{2T}}) + \alpha_{29}\overline{\mathbf{V}}^2\overline{\mathbf{V}}^{\mathbf{T}} + \alpha_{30}\overline{\mathbf{V}}\overline{\mathbf{V}}^{\mathbf{2T}} + \alpha_{31}\overline{\mathbf{V}}^2\overline{\mathbf{V}}^{\mathbf{2T}} + \alpha_{32}[\gamma\gamma^{\mathbf{2T}}(\overline{\mathbf{V}} + \overline{\mathbf{V}}^{\mathbf{T}})\\
&+ \gamma^{\mathbf{2T}}(\overline{\mathbf{V}} + \overline{\mathbf{V}}^{\mathbf{T}})\gamma] + \alpha_{33}\mathbf{I} + \mathbf{2}\alpha_{34}\gamma + \mathbf{3}\alpha_{35}\gamma^2 + \mathbf{2}\alpha_{36}\gamma^{\mathbf{T}} + \alpha_{37}(\gamma\gamma^{\mathbf{T}} + \gamma^{\mathbf{T}}\gamma + \gamma^{\mathbf{2T}})\\
&+ \mathbf{2}\alpha_{38}(\gamma\gamma^{\mathbf{2T}} + \gamma^{\mathbf{2T}}\gamma) + \mathbf{2}\alpha_{39}(\gamma\gamma^{\mathbf{2T}}\gamma\gamma^{\mathbf{T}} + \gamma^{\mathbf{T}}\gamma^2\gamma^{\mathbf{2T}} + \gamma^{\mathbf{2T}}\gamma\gamma^{\mathbf{T}}\gamma)\}.
\end{aligned} \tag{39}$$

Using Cayley–Hamilton theorem, minor simplifications may be made in Eqs. (38) and (39) which are not presented here.

3.4. Incompressiblity of the micropolar media

In the classical continuum theory, the incompressibility condition ($J = \det \mathbf{F} = 1$) results in the pressure term $-p\mathbf{I}$, added to the stress constitutive equation and contributes nothing to the energy or power of deformation. Now, let the traction vector $\mathbf{t}^{(\mathbf{n})} = -p\mathbf{n}$ and the couple vector $\mathbf{m}^{(\mathbf{n})} = -\mathbf{s} \times \mathbf{n}$ (with p and $\mathbf{s}$, respectively, as an arbitrary scalar and an arbitrary vector) contribute nothing to the power of deformation P expressed in Eq. (13). Thus, $\bar{\boldsymbol{\sigma}}$ can be determined to within $-p\mathbf{I}$ and $\mathbf{m}$ to within $-\boldsymbol{\varepsilon}\mathbf{s}$. Substituting $\bar{\sigma} = -p\mathbf{I}$ and $\mathbf{m} = -\boldsymbol{\varepsilon}\mathbf{s}$ into Eq. $(13)_1$, we obtain

$$P = -p\mathbf{I} : \overline{\mathbf{D}} - \boldsymbol{\varepsilon}\mathbf{s} : \boldsymbol{\Psi} = \mathbf{0} \Rightarrow \mathbf{p}\,\mathrm{div}(\mathbf{v}) + \mathbf{s}.\mathrm{curl}(\bar{\omega}) = \mathbf{0}. \tag{40}$$

Since Eq. (40) have to be valid for all p and $\mathbf{s}$, we obtain the following constraints for the incompressibility of a micropolar continuous media (Kafadar and Eringen, 1971):

$$\mathrm{div}(\mathbf{v}) = 0, \quad \mathrm{curl}(\bar{\boldsymbol{\omega}}) = 0. \tag{41}$$

As it is well-known, from $\dot{J} = J\,\mathrm{div}(\mathbf{v})$, the constraint $\mathrm{div}(\mathbf{v}) = 0$ may be integrated to $J = 1$. However, the constraint $\mathrm{curl}(\bar{\boldsymbol{\omega}}) = 0$ implies that $\bar{\boldsymbol{\omega}} = \nabla_x \Phi$, where Φ is a scalar potential function. But such condition, in general, cannot be integrated to impose an explicit restriction on the micro rotation $\boldsymbol{\varphi}$. For the linear theory in which $\bar{\boldsymbol{\omega}} \approx \dot{\boldsymbol{\varphi}}$, this constraint can be integrated to give $\mathrm{curl}(\boldsymbol{\varphi}) = 0$. Finally, the constitutive equations for the micropolar incompressible hyper-elastic materials are as follows (Kafadar and Eringen, 1971):

$$\bar{\boldsymbol{\sigma}} = -p\mathbf{I} + \rho\overline{\mathbf{V}}\frac{\partial\tilde{\psi}(\overline{\mathbf{V}},\gamma)}{\partial\overline{\mathbf{V}}^{\mathbf{T}}}, \quad \mathbf{m} = -\boldsymbol{\varepsilon}\mathbf{s} + \rho\overline{\mathbf{V}}\frac{\partial\tilde{\psi}(\overline{\mathbf{V}},\gamma)}{\partial\gamma^{\mathbf{T}}}. \tag{42}$$

It is noted that the constraints expressed in Eq. (41) have to be imposed on ψ. But, it remains an open problem that how the condition $\mathrm{curl}(\bar{\boldsymbol{\omega}}) = 0$ can be satisfied.

4. Some simple micropolar hyper-elastic constitutive equations

In this section, we present some generalized forms of the Hookean, neo-Hookean, and Mooney-Rivlin type constitutive equations for the micropolar hyper-elastic materials. Except for the case of linear deformations, such generalizations are not unique and one may find several generalizations for a specific type of the micropolar hyper-elastic materials. Furthermore, we will ignore the condition $\mathrm{curl}(\bar{\boldsymbol{\omega}}) = 0$ for the case of micropolar incompressible materials.

4.1. Hookean type micropolar hyper-elasticity (linear micropolar elasticity)

For the case of linear isotropic micropolar materials, the constitutive equations (18) reduce to $\{\bar{\boldsymbol{\sigma}}, \mathbf{m}\} = \rho_0 \partial\,\widehat{\psi}(\bar{\mathbf{e}}, \boldsymbol{\kappa})/\partial\{\bar{\mathbf{e}}, \boldsymbol{\kappa}^T\}$ is as follows (Eringen, 1968):

$$\begin{aligned}\rho_0\,\widehat{\psi}(\bar{\mathbf{e}},\boldsymbol{\kappa}) = {} & \frac{1}{2}[\lambda\,tr(\bar{\mathbf{e}})^2 + (\bar{\mu}+\eta)\,tr(\bar{\mathbf{e}}\bar{\mathbf{e}}^T) + \bar{\mu}\,tr(\bar{\mathbf{e}}^2)] \\ & + \frac{1}{2}[\alpha\,tr(\boldsymbol{\kappa})^2 + \beta\,tr(\boldsymbol{\kappa}^2) + \gamma\,tr(\boldsymbol{\kappa}\boldsymbol{\kappa}^T)],\end{aligned} \tag{43}$$

which results in

$$\begin{aligned}\bar{\boldsymbol{\sigma}} &= \lambda\bar{e}^k_{.k}\mathbf{I} + (\bar{\mu}+\eta)\bar{\mathbf{e}} + \bar{\mu}\bar{\mathbf{e}}^T = \overline{\mathbf{C}} : \bar{\mathbf{e}}, \quad \mathbf{m} = \alpha\kappa^k_{.k}\mathbf{I} + \beta\boldsymbol{\kappa} + \gamma\boldsymbol{\kappa}^T \\ &= \tilde{\mathbf{C}} : \boldsymbol{\kappa},\end{aligned} \tag{44}$$

where λ, $\bar{\mu}$, η, $\bar{\alpha}$, β and γ are elastic constants and the relation $\mu = \bar{\mu} + \eta/2$ holds, with μ as the shear modulus. Also, the micropolar length-scale parameter is defined as $L = \sqrt{\beta/(2\mu)}$ (de Borst, 1993). The fourth-order isotropic elasticity tensors $\overline{\mathbf{C}}$ and $\tilde{\mathbf{C}}$ (or equivalently $\mathbf{a}_1$ and $\mathbf{a}_4$) have the following components ($\mathbf{a}_2 = \mathbf{a}_3 = 0$):

$$\{\overline{C}^{ijkl}, \tilde{C}^{ijkl}\} = \{\lambda, \alpha\}g^{ij}g^{kl} + \{\bar{\mu}+\eta, \beta\}g^{ik}g^{jl} + \{\bar{\mu}, \gamma\}g^{il}g^{jk}. \tag{45}$$

4.2. Neo-Hookean type micropolar hyper-elasticity

In the classical continuum theory, the neo-Hookean type energy function is assumed to be $\rho_0\psi(\mathbf{B}) = \mu(B^n_{.n} - 3)/2 - (p+\mu)\ln(J)|_{J=1}$ which results in $\boldsymbol{\sigma} = \mu(\mathbf{B} - \mathbf{I}) - p\mathbf{I}$. Simo and Pister (1984) extended such energy function to describe compressible materials by defining $\rho_0\psi(\mathbf{B}) = \mu(B^n_{.n} - 3)/2 + \overline{U}(J)$. Here, $\overline{U}(J)$ is proposed to be $\overline{U}(J) = \lambda U(J) - \mu\ln(J)$, which results in $J\boldsymbol{\sigma} = [\lambda J U'(J) - \mu]\mathbf{I} + \mu\mathbf{B}$. Also, $U(J)$ is a convex function satisfying $U(J)|_{J=1} = U'(J)|_{J=1} = 0$ and $\lim_{J\to 0}U(J) = \lim_{J\to\infty}U(J) = \infty$. A suitable example for $U(J)$ is $U(J) = \lambda(\ln J)^2/2$. In order to give a generalization of such model to the micropolar theory, we introduce the following energy function:

$$\begin{aligned}\rho_0\tilde{\psi}(\overline{\mathbf{V}},\gamma) = {} & \frac{\mathbf{1}}{\mathbf{2}}\left\{(\bar{\mu}+\eta)\left[\mathbf{tr}(\overline{\mathbf{V}}\overline{\mathbf{V}}^{\mathbf{T}}) - \mathbf{3}\right] - \frac{\mathbf{1}}{\mathbf{2}}\eta\left[\mathbf{tr}(\overline{\mathbf{V}}^{\mathbf{2}}) - \mathbf{3}\right]\right\} \\ & - \mu\ln(\mathbf{J}) + \lambda\mathbf{U}(\mathbf{J}) + \frac{\mathbf{1}}{\mathbf{2}}\left[\alpha\,\mathbf{tr}(\gamma)^{\mathbf{2}} + \beta\,\mathbf{tr}(\gamma^{\mathbf{2}}) + \gamma\,\mathbf{tr}(\gamma\gamma^{\mathbf{T}})\right],\end{aligned} \tag{46}$$

which renders

$$\bar{\tau} = [\lambda J U'(J) - \mu]\mathbf{I} + (\bar{\mu}+\eta)\mathbf{B} - \frac{1}{2}\eta\overline{\mathbf{V}}^{\mathbf{2}}, \quad \boldsymbol{\mu} = \overline{\mathbf{V}}[\alpha\,\mathbf{tr}(\gamma)\mathbf{I} + \beta\gamma^{\mathbf{T}} + \gamma\gamma]. \tag{47}$$

If we further assume the condition $\lim_{J\to 1}[J\lambda^{-1}U'(J)] = tr\,\mathbf{e}$ on $U(J)$, with $\mathbf{e} = sym(\bar{\mathbf{e}})$ as the classical infinitesimal strain tensor, Eq. (47) reduce to the linear micropolar elasticity (Eq. (44)) for the case of infinitesimal deformations. The function $U(J) = \lambda(\ln J)^2/2$ is an example which satisfies such condition. Furthermore, if $J = 1$, $\eta = 0$ and $\overline{\mathbf{R}} = \mathbf{R}$, classical theory of neo-Hookean materials will be retrieved. Using Eqs. (25) and (46), the tensors $\mathbf{a}_1$ and $\mathbf{a}_4$ take the following forms ($\mathbf{a}_2 = \mathbf{a}_3 = 0$):

$$\begin{aligned}Ja_1^{ijkl} = {} & \{(J^2\lambda U'' + \mu)g^{ij}g^{kl} + (\bar{\mu}+\eta)\overline{V}^i_{.n}\overline{V}^{kn}g^{jl} - \frac{1}{2}\eta\overline{V}^{il}\overline{V}^{kj} + J^{-1}(J\lambda U' - \mu)[\overline{V}^{il}\overline{V}^k_{.n}\overline{V}^{nj} + \overline{V}^{in}\overline{V}^{.l}_{n}\overline{V}^{kj} \\ & - \overline{V}^{in}\overline{V}^{.j}_{n}\overline{V}^{kl} - \overline{V}^n_{.n}\overline{V}^{il}\overline{V}^{kj} + \overline{V}^n_{.n}\overline{V}^{ij}\overline{V}^{kl} - \overline{V}^{ij}\overline{V}^k_{.n}\overline{V}^{nl}]\}, \\ Ja_4^{ijkl} = {} & \alpha\overline{V}^{ij}\overline{V}^{kl} + \beta\overline{V}^{il}\overline{V}^{kj} + \gamma\overline{V}^{im}\overline{V}^k_{.m}g^{jl}.\end{aligned} \tag{48}$$

4.3. Mooney-Rivlin type micropolar hyper-elasticity

A Moony-Rivlin incompressible material, in the classical continuum theory is described by $\rho\psi = \mu\{(\bar{\beta} + 1/2)(B^n_{.n} - 3) + (1/2 - \bar{\beta})[(B^n_{.n}B^p_{.p} - B^n_{.p}B^p_{.n})/2 - 3]\}/2$, where $\bar{\beta}$ is a constant in $[-1/2, 1/2]$. Using Cayley–Hamilton theorem, the resulting stress tensor is $\boldsymbol{\sigma} = \mu[(\bar{\beta} + 1/2)\mathbf{B} - (1/2 - \bar{\beta})\mathbf{B}^{-1}] - (p+\mu)\mathbf{I}$ (Truesdell and Noll, 1965). This energy function may be also extended to the compressible materials by the method proposed by Simo and Pister (1984) or Simo et al. (1985). A generalization to the micropolar media can be expressed by the following energy function:

$$\begin{aligned}\rho_0\tilde{\psi}(\overline{\mathbf{V}},\gamma) = {} & \frac{\mathbf{1}}{\mathbf{2}}\left\{\mathbf{a}\left[\mathbf{tr}(\overline{\mathbf{V}}\overline{\mathbf{V}}^{\mathbf{T}}) - \mathbf{3}\right] + \mathbf{b}\left[\mathbf{tr}(\overline{\mathbf{V}}^{\mathbf{2}}) - \mathbf{3}\right] + \mathbf{c}\left[\mathbf{tr}(\overline{\mathbf{V}}^{\mathbf{2}}\overline{\mathbf{V}}^{\mathbf{2T}})\right.\right. \\ & \left.\left. - (\mathbf{tr}(\overline{\mathbf{V}}^{\mathbf{2}}))^{\mathbf{2}} + \mathbf{6}\right]\right\} + \mathbf{e}\ln(\mathbf{J}) + \mathbf{fU}(\mathbf{J}) \\ & + \frac{\mathbf{1}}{\mathbf{2}}\left[\alpha\,\mathbf{tr}(\gamma)^{\mathbf{2}} + \beta\,\mathbf{tr}(\gamma\gamma^{\mathbf{T}}) + \gamma\,\mathbf{tr}(\gamma^{\mathbf{2}})\right],\end{aligned} \tag{49}$$

where a, b, c, e and f are unknown constants. It is noted that one may add other terms containing the invariants of the wryness tensor γ, but it needs the definition of some new coefficients which we avoid it here. This energy function yields the following stress tensor:

$$\bar{\tau} = [fJU'(J) + e]\mathbf{I} + a\mathbf{B} + b\overline{\mathbf{V}}^{\mathbf{2}} + \mathbf{c}\left[\overline{\mathbf{V}}\mathbf{B}\overline{\mathbf{V}}^{\mathbf{T}} + \mathbf{B}\overline{\mathbf{V}}^{\mathbf{T}}\overline{\mathbf{V}} - \mathbf{2}\,\mathbf{tr}(\overline{\mathbf{V}}^{\mathbf{2}})\overline{\mathbf{V}}^{\mathbf{2}}\right] \tag{50}$$

and $\boldsymbol{\mu}$ is the same as Eq. $(47)_2$. Now, we assume that when $J = 1$, $\eta = 0$ and $\overline{\mathbf{R}} = \mathbf{R}$, Eq. $(50)_1$ can be reduced to the Cauchy stress $\boldsymbol{\sigma}$

in the classical Mooney-Rivlin materials. Also, we assume that when the deformations are infinitesimal and the condition $\lim_{J\to 1}[J\lambda^{-1}U'(J)] = tr\,\mathbf{e}$ holds, the expression for $\bar{\tau}$ can be reduced to the micropolar linear elastic response. Under these circumstances, the unknown coefficients are obtained as follows:

$$a = 2\mu(1-\bar{\beta}) + \frac{1}{2}\eta,\quad b = 3\mu\left(\bar{\beta}-\frac{1}{2}\right) - \frac{1}{2}\eta,\quad c = -\frac{1}{2}\mu\left(\frac{1}{2}-\bar{\beta}\right),$$
$$e = \mu\left(\bar{\beta}-\frac{3}{2}\right),\quad f = -2\mu\left(\frac{1}{2}-\bar{\beta}\right)+\lambda. \tag{51}$$

Now, using Eqs. (25) and (49), $\mathbf{a}_1$ takes the following forms ($\mathbf{a}_4$ is the same as Eq. $(48)_2$ and again $\mathbf{a}_2 = \mathbf{a}_3 = 0$):

$$\begin{aligned} Ja_1^{ijkl} =& (fJ^2U'' - e)g^{ij}g^{kl} + a\overline{V}^{i}_{.n}\overline{V}^{kn}g^{jl} + b\overline{V}^{il}\overline{V}^{kj} \\ &+ J^{-1}(fJU' + e)(\overline{V}^{il}\overline{V}^{k}_{.n}\overline{V}^{nj} + \overline{V}^{im}\overline{V}_{m}^{.l}\overline{V}^{kj} - \overline{V}^{im}\overline{V}_{m}^{.j}\overline{V}^{kl} \\ &- \overline{V}^{n}_{.n}\overline{V}^{il}\overline{V}^{kj} + \overline{V}^{n}_{.n}\overline{V}^{ij}\overline{V}^{kl} - \overline{V}^{ij}\overline{V}^{k}_{.n}\overline{V}^{nl}) + c(\overline{V}^{il}\overline{V}^{k}_{.p}\overline{V}_{q}^{.p}\overline{V}^{qj} \\ &+ \overline{V}^{im}\overline{V}_{m}^{.n}\overline{V}^{k}_{.n}\overline{V}^{jl} + \overline{V}^{im}\overline{V}_{m}^{.p}\overline{V}^{n}_{.p}\overline{V}^{k}_{.n}g^{jl} + \overline{V}^{im}\overline{V}^{k}_{.m}\overline{V}^{l}_{.q}\overline{V}^{qj} \\ &+ \overline{V}^{im}\overline{V}^{n}_{.m}\overline{V}^{k}_{.n}\overline{V}^{lj} + \overline{V}^{im}\overline{V}^{p}_{.m}\overline{V}^{l}_{.p}\overline{V}^{kj} - 4\overline{V}^{im}\overline{V}_{m}^{.j}\overline{V}^{kp}\overline{V}_{p}^{.l} \\ &- 2\overline{V}^{n}_{.n}\overline{V}^{m}_{.m}\overline{V}^{il}\overline{V}^{kj}). \end{aligned} \tag{52}$$

It is noted that the multiplicative decomposition of $\overline{\mathbf{V}}$ in the form $\overline{\mathbf{V}} = \mathbf{J}^{1/3}\hat{\mathbf{V}}$ may be used for better and more logical formulation of the proposed constitutive equations. This decomposition is based on the fundamental decomposition $\mathbf{F} = J^{1/3}\hat{\mathbf{F}}$, which has been frequently employed in the literature of the classical continuum theory (see e.g. Simo et al., 1985). However, such formulations in the context of the micropolar theory and especially for the energy functions proposed in Eqs. (46) and (49), results in very complex expressions for the tensors $\bar{\tau}$, $\boldsymbol{\mu}$ and specially $\mathbf{a}_1$ and $\mathbf{a}_4$ which are not presented here.

5. Updated Lagrangian finite element formulation

In this section, Updated Lagrangian FE formulations based on the general form of the constitutive equations for the isotropic micropolar hyper-elastic materials (26) is derived. A Cartesian coordinate system and equilibrium form of Eqs. (11) and (12) are employed. Suppose, we are in the Υth load step ($\Upsilon = 1, 2, \ldots, \overline{\Upsilon}$ with $\overline{\Upsilon}$ as an arbitrary natural number); the solution for the Σth sub-step has been found but the convergence criterion has not yet been satisfied, thus we should solve the problem for the $(\Sigma+1)$th sub-step. We consider a typical element with n_e nodes having the translational and m_e nodes with rotational degrees of freedom (DOFs) in which some or all nodes of the element can have both translational and rotational DOFs. We consider the following FE-approximations:

$$v_i(\mathbf{x},t) = N_\upsilon(\mathbf{x})V_{\upsilon i}(t),\quad \dot{\varphi}_j(\mathbf{x},t) = M_\vartheta(\mathbf{x})\Theta_{\vartheta j}(t),$$
$$\upsilon = 1,2,.,n_e,\quad \vartheta = 1,2,.,m_e \tag{53}$$

where v_i, $\dot{\varphi}_j$, N_υ, M_ϑ, $V_{\upsilon i}$ and $\Theta_{\vartheta j}$ are the velocity component in i-direction, rate of rotation of micro-structure about j-axis, interpolation functions for velocity, interpolation functions for the rate of micro-rotation $\dot{\boldsymbol{\varphi}}$, nodal velocity of the υth node in the i-direction and nodal rate of rotation of the ϑth node about j-axis, respectively. Eq. (53) yield the following expressions for $\mathbf{L}$, $\overline{\mathbf{D}}$, $\overline{\boldsymbol{\Omega}}$ and $\boldsymbol{\Psi}$:

$$\begin{aligned} &L_{ij} = \frac{\partial v_i}{\partial x_j} = \frac{\partial N_\upsilon}{\partial x_j}V_{\upsilon i},\quad \overline{D}_{ij} = \frac{\partial N_\upsilon}{\partial x_i}V_{\upsilon j} - \varepsilon_{ijk}\Lambda_{km}M_\vartheta\Theta_{\vartheta m}, \\ &\overline{\Omega}_{ij} = -\varepsilon_{ijk}\Lambda_{km}M_\vartheta\Theta_{\vartheta m},\quad \Psi_{ij} = \left(\overline{\Lambda}_{jmi}M_\vartheta + \Lambda_{jm}\frac{\partial M_\vartheta}{\partial x_i}\right)\Theta_{\vartheta m}, \end{aligned} \tag{54}$$

where in Eq. $(54)_4$, the relation $\overline{\boldsymbol{\Lambda}} = \nabla_x\boldsymbol{\Lambda}$ holds.

In order to discretize the linear momentum equations, we pre-multiply Eq. (11) by N_α ($\alpha = 1,2,.,n_e$) and integrate over the volume of a typical element to obtain

$$\Pi_{\alpha i} = f_{\alpha i}^{int} - f_{\alpha i}^{ext},\quad \alpha = 1,2,\ldots,n_e, \tag{55}$$

where $\Pi_{\alpha i}$ are components of the residual matrix, and $f_{\alpha i}^{int}$ and $f_{\alpha i}^{ext}$ are components of the internal and external nodal force matrices, respectively, with the following expressions:

$$f_{\alpha i}^{int} = \int_{\mathscr{R}_0}\frac{\partial N_\alpha}{\partial X_J}P_{Ji}\,\mathrm{d}v_0,\quad f_{\alpha i}^{ext} = \int_{\mathscr{R}_0}\rho N_\alpha f_i\,\mathrm{d}v + \oint_{\mathscr{S}}N_\alpha t_i\,\mathrm{d}a, \tag{56}$$

where $\mathscr{R}_0$, $\mathscr{R}$, $\mathrm{d}v_0$ and $\mathrm{d}v$ are the volume region and volume element of the typical element in the initial and current configurations, respectively. Also, $\mathscr{S}$ and $\mathrm{d}a$ are the boundary surface and area of the typical element in the current configuration. Applying the standard Newton–Raphson method (Belytschko et al., 2000) and for the case in which the external load is not follower ($\dot{f}_{\alpha i}^{ext} = 0$), we obtain

$$\left(K_{\alpha i\upsilon q}^{uu}\right)_\Upsilon^\Sigma(V_{\upsilon q}\Delta t)_\Upsilon^\Sigma + \left(K_{\alpha i\vartheta q}^{u\theta}\right)_\Upsilon^\Sigma(\Theta_{\vartheta q}\Delta t)_\Upsilon^\Sigma = -(\Pi_{\alpha i})_\Upsilon^\Sigma, \tag{57}$$

$$\left(K_{\alpha i\upsilon q}^{uu}\right)_\Upsilon^\Sigma = \left(\frac{\partial\dot{f}_{\alpha i}^{int}}{\partial V_{\upsilon q}}\right)_\Upsilon^\Sigma,\quad \left(K_{\alpha i\vartheta q}^{u\theta}\right)_\Upsilon^\Sigma = \left(\frac{\partial\dot{f}_{\alpha i}^{int}}{\partial\Theta_{\vartheta q}}\right)_\Upsilon^\Sigma, \tag{58}$$

where $()_\Upsilon^\Sigma$ stands for the underlying quantity at Υth load step and Σth sub-step. Also, $K_{\alpha i\upsilon q}^{uu}$ and $K_{\alpha i\vartheta q}^{u\theta}$ are components of the stiffness matrices $\mathbf{K}^{uu}$ and $\mathbf{K}^{u\theta}$ corresponding to the linear momentum equations. For later use, we define $\overline{\mathbf{B}}$ **and** $\tilde{\mathbf{B}}$ matrices as follows:

$$\overline{B}_{\upsilon i} = \frac{\partial N_\upsilon}{\partial x_i},\quad \tilde{B}_{\vartheta i} = \frac{\partial M_\vartheta}{\partial x_i},\quad \upsilon = 1,2,\ldots,n_e,\ \ \vartheta = 1,2,\ldots,m_e. \tag{59}$$

By $\bar{\boldsymbol{\sigma}}^\nabla + \mathbf{L}\bar{\boldsymbol{\sigma}} - \bar{\boldsymbol{\sigma}}\overline{\boldsymbol{\Omega}} - tr(\mathbf{D})\bar{\boldsymbol{\sigma}}$ and Eq. (26), the components of $\dot{f}_{\alpha i}^{int}$ are as follows:

$$\dot{f}_{\alpha i}^{int} = \int_{\mathscr{R}}\overline{B}_{\alpha j}[a_{1jipq}\overline{D}_{pq} + a_{2jipq}\Psi_{pq} + \bar{\sigma}_{jn}\overline{\Omega}_{ni}]\mathrm{d}v. \tag{60}$$

Now, using Eqs. (54) and (58), we obtain

$$K_{\alpha i\upsilon q}^{uu} = \int_{\mathscr{R}}\overline{B}_{\alpha j}a_{1jipq}\overline{B}_{\upsilon p},\mathrm{d}v, \tag{61}$$

$$K_{\alpha i\vartheta q}^{u\theta} = \int_{\mathscr{R}}\overline{B}_{\alpha j}[-\varepsilon_{pms}M_\vartheta\Lambda_{sq}a_{1jipm} + a_{2jipm}(M_\vartheta\overline{\Lambda}_{mqp} + \tilde{B}_{\vartheta p}\Lambda_{mq}) + \varepsilon_{nim}M_\vartheta\Lambda_{mq}\bar{\sigma}_{jn}]\mathrm{d}v. \tag{62}$$

The symbol $()_\Upsilon^\Sigma$ has been dropped for simplicity and all the integrals have to be taken over the volume region and surface area of the typical element at Σth sub-step of the Υth load-step, i.e., $\mathscr{R}_\Upsilon^\Sigma$ and $\mathscr{S}_\Upsilon^\Sigma$.

Now, we discretize the moment of momentum equations by pre-multiplying Eq. (12) by M_β ($\beta = 1,2,\ldots,m_e$) and integrating over the volume of a typical element which gives

$$\overline{\Pi}_{\beta i} = m_{\beta i}^{int} - m_{\beta i}^{ext},\quad \beta = 1,2,\ldots,m_e, \tag{63}$$

where $\overline{\Pi}_{\beta i}$ are components of a residual matrix, $m_{\beta i}^{int}$ and $m_{\beta i}^{ext}$ are, respectively, the components of the internal and external nodal couple stress matrices as follows:

$$m_{\beta i}^{int} = \int_{\mathscr{R}_0}\left[\frac{\partial M_\beta}{\partial X_J}M_{Ji} - \varepsilon_{jmi}M_\beta F_{jK}P_{Km}\right]\mathrm{d}v_0, \tag{64}$$

$$m_{\beta i}^{ext} = \int_{\mathscr{R}}\rho M_\beta l_i\,\mathrm{d}v + \oint_S M_\beta m_i\,\mathrm{d}a. \tag{65}$$

Following standard Newton–Raphson method and for the case of no follower loading condition, we obtain the following linearized equations:

$$\left(K_{\beta i\upsilon q}^{\theta u}\right)_\Upsilon^\Sigma(V_{\upsilon q}\Delta t)_\Upsilon^\Sigma + \left(K_{\beta i\vartheta q}^{\theta\theta}\right)_\Upsilon^\Sigma(\Theta_{\vartheta q}\Delta t)_\Upsilon^\Sigma = -(\overline{\Pi}_{\beta i})_\Upsilon^\Sigma, \tag{66}$$

$$\left(K_{\beta i\upsilon q}^{\theta u}\right)_\Upsilon^\Sigma = \left(\frac{\partial\dot{m}_{\beta i}^{int}}{\partial V_{\upsilon q}}\right)_\Upsilon^\Sigma,\quad \left(K_{\beta i\vartheta q}^{\theta\theta}\right)_\Upsilon^\Sigma = \left(\frac{\partial\dot{m}_{\beta i}^{int}}{\partial\Theta_{\vartheta q}}\right)_\Upsilon^\Sigma. \tag{67}$$

Here, $K^{\theta u}_{\beta i \upsilon q}$ and $K^{\theta\theta}_{\beta i \vartheta q}$ are components of the stiffness matrices $\mathbf{K}^{\theta u}$ and $\mathbf{K}^{\theta\theta}$ corresponding to the moment of momentum equations. A straight forward calculation yields

$$\dot{m}^{int}_{\beta i} = \int_{\mathscr{R}} \Big\{ \tilde{B}_{\beta j}[a_{3jipm}\overline{D}_{pm} + a_{4jipm}\Psi_{pm} - m_{jn}\overline{\Omega}_{ni}] - \varepsilon_{ijn}M_{\beta}[a_{1jnpm}\overline{D}_{pm} + a_{2jnpm}\Psi_{pm} + L_{jp}\bar{\sigma}_{pn} - \bar{\sigma}_{jp}\bar{\Omega}_{pn}] \Big\} \mathrm{d}v, \tag{68}$$

which results in

$$K^{\theta u}_{\beta i \upsilon q} = \int_{\mathscr{R}} [\tilde{B}_{\beta j}a_{3jipq}\overline{B}_{\upsilon p} - \varepsilon_{ijn}M_{\beta}\overline{B}_{\upsilon p}a_{1jnpq} - \varepsilon_{iqn}M_{\beta}\overline{B}_{\upsilon j}\bar{\sigma}_{jn}]\mathrm{d}v, \tag{69}$$

$$K^{\theta\theta}_{\beta i \vartheta q} = \int_{\mathscr{R}} \{\tilde{B}_{\beta j}[-\varepsilon_{pmk}a_{3jipm}\Lambda_{kq}M_{\vartheta} + a_{4jipm}(\overline{\Lambda}_{mqp}M_{\vartheta} + \Lambda_{mq}\tilde{B}_{\vartheta p})] - \varepsilon_{ijn}M_{\beta}a_{2jnpm}(\overline{\Lambda}_{mqp}M_{\vartheta} + \Lambda_{mq}\tilde{B}_{\vartheta p}) - \varepsilon_{ijn}\varepsilon_{pnk}\Lambda_{kq}\bar{\sigma}_{jp} + \varepsilon_{ijn}\varepsilon_{pmk}M_{\beta}a_{1jnpm}\Lambda_{kq}M_{\vartheta}\}\mathrm{d}v. \tag{70}$$

After calculating the desired stiffness and residual matrices, finally we arrive at the following matrix equation for finding the unknowns in the $(\Sigma+1)$th sub-step:

$$\mathbf{K}^{\Sigma}_{\Upsilon}\Delta\overline{\mathbf{V}}^{\Sigma}_{\Upsilon} = -\tilde{\mathbf{\Pi}}^{\Sigma}_{\Upsilon}, \tag{71}$$

where the stiffness matrix $\mathbf{K}$, the generalized displacement matrix $\Delta\overline{\mathbf{V}}$, and the generalized residual matrix $\tilde{\mathbf{\Pi}}$ are defined as

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}^{uu} & \mathbf{K}^{u\theta} \\ \mathbf{K}^{\theta u} & \mathbf{K}^{\theta\theta} \end{bmatrix}_{(n_e+m_e)\times(n_e+m_e)}, \quad \Delta\overline{\mathbf{V}} = \begin{bmatrix} \mathbf{V}\Delta t \\ \mathbf{\Theta}\Delta t \end{bmatrix}_{(\mathbf{n_e}+\mathbf{m_e})\times 3}, \quad \tilde{\mathbf{\Pi}} = \begin{bmatrix} \mathbf{\Pi} \\ \overline{\mathbf{\Pi}} \end{bmatrix}_{(\mathbf{n_e}+\mathbf{m_e})\times 3}. \tag{72}$$

The geometry of the $(\Sigma+1)$th sub-step is updated by $\mathbf{u}^{\Sigma+1}_{\Upsilon} = \mathbf{u}^{\Sigma}_{\Upsilon} + \Delta\mathbf{u}^{\Sigma}_{\Upsilon}$ at each node of the body, while the micro-rotation $\boldsymbol{\varphi}$ is updated through the following relations (Steinmann, 1994):

$$\bar{\boldsymbol{\varphi}} = \frac{\boldsymbol{\varphi}}{\theta}\tan\frac{\theta}{2}, \quad \bar{\boldsymbol{\varphi}}^{\Sigma+1}_{\Upsilon} = \left[\frac{\bar{\boldsymbol{\varphi}} + \Delta\bar{\boldsymbol{\varphi}} - (\boldsymbol{\varepsilon}.\Delta\bar{\boldsymbol{\varphi}})\bar{\boldsymbol{\varphi}}}{1 - \bar{\boldsymbol{\varphi}}.\Delta\bar{\boldsymbol{\varphi}}}\right]^{\Sigma}_{\Upsilon}, \quad \boldsymbol{\varphi}^{\Sigma+1}_{\Upsilon} = \left[\frac{2\tan^{-1}\bar{\theta}}{\bar{\theta}}\bar{\boldsymbol{\varphi}}\right]^{\Sigma+1}_{\Upsilon}. \tag{73}$$

Furthermore, since the order of magnitude of micro-rotation is often smaller than that of macro-displacement, we may use two different convergence criterions for them.

6. Numerical examples

In this section, we present two planar examples solved by our FE formulations. It is noted that after some minor simplifications of the general 3D FE formulations obtained in the previous section, they can be applied to 2D problems. For this purpose, a finite element program has been prepared and a 9-node element, with 24 DOFs, 18 translational and 4 rotational ones has been used for discretizing the geometry. All 9-nodes of the element have two translational DOFs (u_x, u_y) and four nodes located at four apexes of the element have micro-rotational DOF (φ_z) beside translational DOFs.

6.1. Bending of a cantilever beam

In this example, a cantilever beam is modeled as a slender rectangle in plane stress condition and is subjected to non-follower distributed loading as shown in Fig. 1. The classical material properties are $E = 1.2\times10^4$ psi and $\upsilon = 0.2$ in all analyses. We do not consider fixed values for the micropolar material parameters and specify them in each diagram. Also, in all analyses, 10 elements along the beam length are considered. The solutions are obtained using 10 load steps $(\overline{\Upsilon} = 10)$ with 3–5 iterations per step.

At first, we consider a beam with the ordinary dimensions (a macro-beam) $B = 10$ in., $H = 1$ in., $W = 1$ in. as it has been considered by Bathe et al. (1975). The compressible neo-Hookean type constitutive equations presented in (47) and (48) with $U(J) = \lambda(\ln J)^2/2$ have been employed. Variations of the deflection ratio U_y/B versus the load parameter $K = PB^3/(EI)$ at tip of the beam for $\eta = G/100$ and different values of the length scale parameter L are demonstrated in Fig. 2. The parameter U_y denotes displacement in the y-direction and I is the area moment of inertia of the beam cross-section about its neutral axis, where the definition of the neutral axis is the same as the classical continuum theory. In order to validate our results, we compare our analysis for the cases where the micropolar effects are negligible, i.e., by setting $\eta = G/100$ and $L = 1\times10^{-6}$ in. In this case, we see that our results are very close to the results of Bathe et al. (1975) in the context of the classical continuum theory. When L increases, the curves pertaining to the micropolar theory departure from that of the classical theory and the deflection of the beam decreases. This means that an increase in L, results in activation of the couple stresses and the resistance of the beam for deflection increases. But, an important point is that for this beam, micropolar effects rise at almost large values of L.

In the next step, we consider a beam with micro-inch dimensions (a micro-beam), i.e., $B = 10\times10^{-6}$ in., $H = 1\times10^{-6}$ in. and $W = 1\times10^{-6}$ in. and with the same values for E and ν. Variations of U_y/B versus K at tip of the beam for different values of L and the shear parameter $a = G/\eta$ are shown in Fig. 3. The patterns of these curves are the same as those shown in Fig. 2, but it is very important to note that the dependence of these curves on the length-scale parameter occurs in very small values of L. In other words, a value of, e.g. $L = 3\times10^{-7}$ in. has not any sensible effect on the deformation of the macro-beam, while this value has significant effects on the deformation of the micro-beam considered here. This means that when the dimensions of a specimen are small, e.g. in micro-level, micropolar effects have an important role in predicting the behavior of materials. Also, this figure shows that when the micropolar shear parameter η increases (or a decreases), a reduction in the deflection of the system can be observed.

6.2. Uniaxial tension of a sheet with a hole

In this example, a square rubber sheet with a half length $B = 10$ in. including a central hole with 3 in. radius in the plane stress condition and under uniaxial distributed loading is examined (Fig. 4). Only one quarter of the sheet has been analyzed and the pattern of elements is similar to those of Bathe et al. (1975). The classical material parameters are $\nu = 0.49$, $\mu = 64$ psi and $\bar{\beta} = 0.281$. The total external load is $p = 50$ lb/in. Variations of the external force parameter $P = pB/6$ versus the horizontal displacement of point A analyzed by different constitutive equations have been shown in Fig. 5. When the micropolar parameters are small, the results of the micropolar neo-Hookean and Mooney-Rivlin constitutive equations are very similar to the classical analysis obtained by Bathe et al. (1975). However, in this specific example, when the micropolar parameters increase, no any significant change of the response is observed. Separation of the classical and micropolar responses occurs at very large values of the length scale parameter, e.g., $L \approx 500$ in. A decrease in the dimensions of the problem, e.g., $B = 10\times10^{-6}$ in. has also no any significant effect on such behavior. Also, changing the classical material parameters, e.g., an increase in the shear modulus μ or decreasing the Poisson ratio associated with the micropolar parameters, do not change the trend of these curves. Hence, we conclude that the influence of the micropolar parameters is dependent not only on the size, but also on the geometry and loading conditions of the body. For the problems in which the deformation is very close to a homogeneous state, the micropolar mechanisms are not acti-

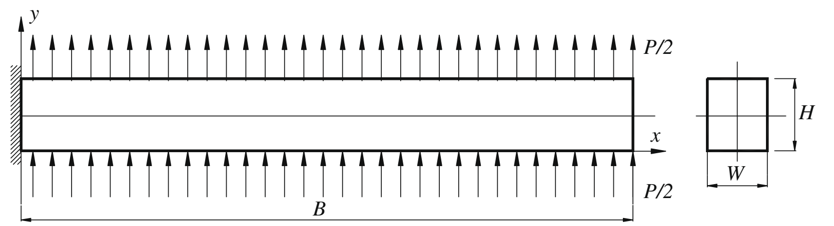

Fig. 1. Geometry and loading of a cantilever beam subjected to a fixed direction distributed load on the upper and lower surfaces.

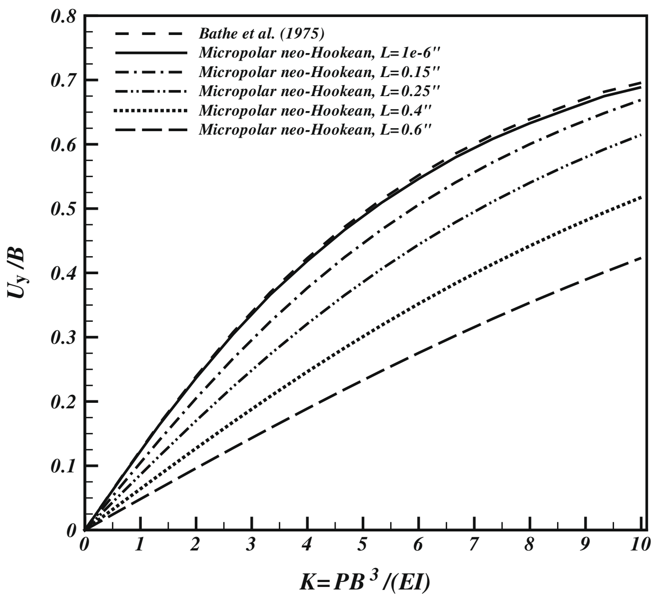

Fig. 2. Variation of deflection ratio U_y/B versus the load parameter $K = PB^3/EI$ at tip of the macro-beam for $a = G/\eta = 100$ and different values of the length scale parameter L.

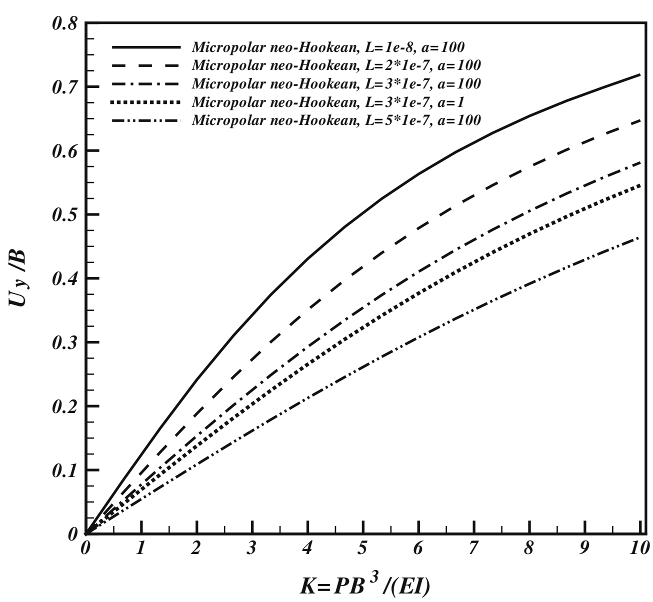

Fig. 3. Variation of deflection ratio U_y/B versus the load parameter $K = PB^3/EI$ at tip of the micro-beam for different values of L and a.

vated. In this case, the large values of length scale parameter or micropolar shear parameter have not any effect on the deformation of the body.

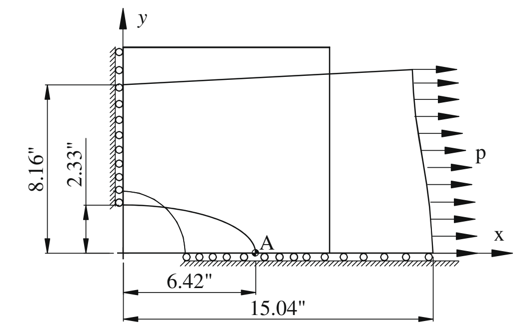

Fig. 4. Geometry, loading and deformed shape of one quarter of a rubber sheet under uniaxial distributed loading for $L = 1 \times 10^{-6}$ in. and $a = 100$, $p = 50$ lb/in..

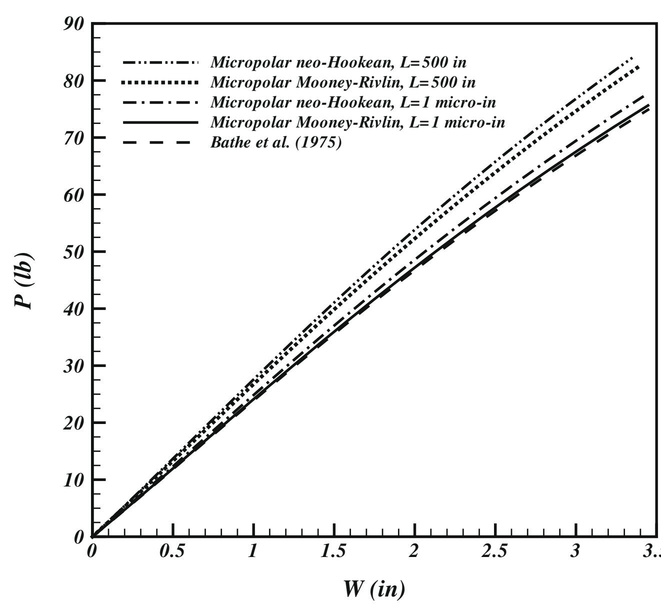

Fig. 5. Variations of the external force $P = pB/6$ versus the displacement of point A.

7. Conclusions

In this paper, hyper-elastic type constitutive equations in the micropolar continuum theory were investigated. Using the representation theorems of tensor functions, the general form of the stress and couple stress tensors based on a general form of the Helmholtz energy function in a purely mechanical process were obtained. The rate-form of the constitutive equations with material and spatial descriptions were derived. The necessary and sufficient conditions for the integrability of the constitutive equations for pre-determined material and spatial elasticities were investigated. As some special cases, the constitutive equations of neo-Hookean and Mooney-Rivlin types were generalized to the micropolar continuum theory. The generalized constitutive equations reduce to those of the linear microplar theory in the case of small deformations. Also, updated Lagrangian finite element formulations based on the general form of the isotropic micropolar hyper-elastic materials were presented. Considering two planar examples, it was shown that an increase in the micropolar parameter results in the stiffening of the body and reduction of its deformation. Also, it was shown that for a specimen with micro-inch dimensions, the micropolar effects are more sensible. This is the so-called conclusion that the smaller is the stiffer. Furthermore, it was shown that the influence of the micropolar parameters is dependent not only on the size of the body, but also to its geometry and loading conditions. For the problems in which the pattern of deformation is very close to a homogeneous state, the micropolar effects are negligible. In these circumstances, classical continuum theory is capable for predicting the behavior of materials.

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